Our goal: Construct this left adjoint.

Def R ring, M right R-module, N left R-module

$F_{(M,N)}$ - free abelian group with basis $M \times N$, i.e.

$$F_{(M,N)} = \left\{ \sum_{\substack{\text{finite}\\\text{sum}}} a_i (m_i, n_i) \mid a_i \in \mathbb{Z}, m_i \in M, n_i \in N \right\}$$

The tensor product of M and N, denoted $M \otimes_R N$, is the abelian group defined as follow

$$M \otimes_R N = \frac{F_{(M,N)}}{\left\langle \begin{array}{l} (m, n_1 + n_2) - (m, n_1) - (m, n_2), \forall m \in M, n_1, n_2 \in N \\ (m_1 + m_2, n) - (m_1, n) - (m_2, n), \forall m_1, m_2 \in M, n \in N \\ (m, r \cdot n) - (m \cdot r, n), \forall m \in M, n \in N, r \in R \end{array} \right.}$$

Denote the image of basis elt $(m,n) \in F$ in $M \otimes_R N$ by $m \otimes n$, called elementary tensor

Note: • elements in $M \otimes_R N$ are finite sums $\sum m_i \otimes n_i$, and may not be given by just a single elementary tensor

- $0_R \otimes n = 0_{M \otimes_R N} = m \otimes 0_N$, Can have $m \otimes_R n = 0$ but $m \neq 0$, $n \neq 0$
- Can have $M \otimes_R N = (0)$ but $M \neq (0)$, $N \neq (0)$

For example $\mathbb{Z}/(2) \otimes \mathbb{Z}/(3)$: $(a+(2)) \otimes (b+(3))$

$= (a+(2)) \otimes 2(2b+(3))$

$= (a+(2)) \cdot 2 \otimes (2b+(3)) = 0 \otimes (2b+3) = 0$

Lemma $M \in \overline{\text{Mod}\, R}$. Have isomorphism

$R \otimes_R M \cong M$.

Pf Define map of abelian groups

$$F_{(R,M)} \longrightarrow M$$

$$\sum_i a_i (r_i, m_i) \longmapsto \sum_i r_i m_i$$

Clearly descends to additive map

$$R \otimes_R M \longrightarrow M$$

$$r \otimes m \longmapsto r \cdot m$$

This has an inverse given by

$$\begin{aligned} M &\longrightarrow R \otimes_R M \\ m &\longmapsto 1 \otimes m \end{aligned} \quad \text{(check!)} \quad \blacksquare$$

Have canonical map

$$\begin{aligned} M \times N &\xrightarrow{\phi} M \otimes_R N \\ m \times n &\longmapsto m \otimes n \end{aligned}$$

This satisfies a universal property:

<u>Def</u>: M right R-module, N left R-module, G abelian group.

A map $f: M \times N \to G$ is <u>R-balanced</u> if

$f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n) \quad \forall m_1, m_2 \in M, n \in R$

$f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2) \quad \forall m \in M, n_1, n_2 \in N$

$f(mr, n) = f(m, rn) \quad \forall m \in M, n \in N, r \in R$

<u>Theorem</u> M right R-module, N left R-module.

The following hold:

(i) The map $\phi: M \times N \to M \otimes_R N$ is R-balanced.

(2) If G abelian group and
$f: M \times N \to G$ R-balanced map,
then exist a unique morphism of
abelian groups $\bar{f}: M \otimes_R N \to G$ s.t.
$f = \bar{f} \circ \phi$

$$\begin{tikzcd} M \times N \arrow[d, "\phi"'] \arrow[dr, "f"] & \\ M \otimes_R N \arrow[r, dashed, "\exists ! \bar{f}"'] & G \end{tikzcd}$$

Proof: Exercise ∎

$f: N_1 \to N_2$ morphism of left R-modules
M right R-module
Then exists unique map
$id \otimes f: M \otimes_R N_1 \to M \otimes_R N_2$
s.t.

$$\begin{tikzcd} M \times N_1 \arrow[r, "id \times f"] \arrow[d] & M \times N_2 \arrow[d] \\ M \otimes_R N_1 \arrow[r, "id \otimes f"] & M \otimes_R N_2 \end{tikzcd}$$

commutes. (use that $M \times N_1 \xrightarrow{id \times f} M \times N_2 \to M \otimes_R N_2$
is R-balanced.

explicitely, $id \otimes f(\sum m_i \otimes n_i) = \sum m_i \otimes f(n_i)$

Hence $N \longmapsto M \otimes_R N$ induces a functor

$M \otimes_R - : \text{Mod}\, R \longrightarrow Ab.$

Similarly, have functor

$- \otimes_R N : \underset{\uparrow \text{right } R\text{-modules}}{\text{Mod}(R^{op})} \longrightarrow Ab$ (N left R-module)

Now if ${}_S M_R$ is an S-R bimodule, then

$M \otimes_R N$ is a left S-module via

$s \cdot (m \otimes n) = (sm) \otimes n$

In fact, can show $M \otimes_R -$ induces a functor

$M \otimes_R - : \text{Mod}\, R \longrightarrow \text{Mod}\, S.$

This is the left adjoint to the Hom-functor!

$\text{Hom}_S(M, -) : \text{Mod}\, S \longrightarrow \text{Mod}\, R.$

<u>Theorem</u>: Let N left R-module, ${}_S M_R$ S-R bimodule, K left S-module. Then have isomorphism

$$\mathrm{Hom}_S(M \otimes_R N, K) \xrightarrow{\cong} \mathrm{Hom}_R(N, \mathrm{Hom}_S(M, K))$$

natural in M, N and K

<u>Pf</u>: Have mutual inverse maps

$$\mathrm{Hom}_S(M \otimes_R N, K) \longrightarrow \mathrm{Hom}_R(N, \mathrm{Hom}_S(M, K))$$

$$\varphi \longmapsto \hat{\varphi}: N \to \mathrm{Hom}_S(M, K)$$

s.t. $n \in N$, $\hat{\varphi}(n): M \to K$ where $\hat{\varphi}(n)(m) = \varphi(m \otimes n)$

$$\tilde{\psi}: M \otimes_R N \to K \longleftarrow\!\shortmid \psi: N \to \mathrm{Hom}_S(M, K)$$

$\tilde{\psi}(m \otimes n) = \psi(n)(m)$

(check they are well-defined!)