$\mathrm{Ext}^{n+1}(A,B)=0 \ \forall A,B\in\mathcal{A}$.

Write $\mathrm{gl.dim}\,\mathcal{A}=\infty$ if no such integer exists.

<u>Theorem</u> $\mathcal{A}$ abelian cat with enough projectives and enough injectives. Let $n\geq 0$ be an integer. The following are equivalent:

(1) $\mathrm{gl.dim}\,\mathcal{A} \leq n$

(2) $\mathrm{pd}\,A \leq n \ \forall A\in\mathcal{A}$

(3) $\mathrm{id}\,A \leq n \ \forall A\in\mathcal{A}$

(4) $\mathrm{Ext}^i_{\mathcal{A}}(A,B)=0 \ \forall i\geq n+1, \forall A,B\in\mathcal{A}$.

<u>Proof</u>: This follows from the previous theorem and its dual ∎

<u>Examples</u>:

(1) $\mathbb{K}$ field, then $\mathrm{gl.dim}\,\mathrm{Mod}\,\mathbb{K}=0$

(2) $\mathrm{gl.dim}\,\mathrm{Mod}\,\mathbb{Z}=1$

$\mathrm{gl.dim}\,\mathrm{Mod}\,\mathbb{K}[x]=1$

More generally, R principal ideal domain (comm ring, no $\neq 0$ zero divisor, every ideal gen by a single elt).

Then gl.dim Mod $R = 1$

(3) gl.dim Mod $\mathbb{k}[x_1, \ldots, x_n] = n$.

<u>Proposition</u> $\mathcal{A}$ abelian cat with enough proj or enough inj, and let $A, C \in \mathcal{A}$. If $\mathrm{Ext}^1_{\mathcal{A}}(C, A) = 0$, then any short exact sequence $0 \to A \to B \to C \to 0$ is split.

<u>Proof</u>: Assume $\mathcal{A}$ has enough projectives (enough injectives is dual).

Choose an exact sequence

$0 \to \Omega C \to P \to C \to 0$ with P projective.

For any exact sequence $0 \to A \to B \to C \to 0$, can find comm diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega C \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

since P is projective.

Since left hand square is a pushout,

$0 \to A \to B \to C \to 0$ is split

$\iff \Omega C \to A$ factors through $\Omega C \to P$ (check!)

By dimension shifting, have

$\mathrm{Ext}^1(C, A) \cong \mathrm{Coker}(\mathrm{Hom}(P, A) \to \mathrm{Hom}(\Omega C, A))$

Hence $\mathrm{Hom}(P, A) \to \mathrm{Hom}(\Omega C, A)$ is surjective,

so any morphism $\Omega C \to A$ factors through $\Omega C \rightarrowtail P$ ▼

Triangulated categories $\mathcal{A}$ additive.

$K(\mathcal{A})$ <u>not</u> abelian in general.
Does it have more properties/structure than just additivity?
Yes! It is a triangulated category.

<u>Def</u>: A <u>triangulated category</u> is a tuple $(\mathcal{T}, [1], \Delta)$ where

- $\mathcal{T}$ - additive category
- $[1] : \mathcal{T} \to \mathcal{T}$ autoequivalence
- Δ - class of sequences of the form
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

s.t.

(T1) - $\forall$ morphisms $f : X \to Y$
$$\exists \left(X \xrightarrow{f} Y \to Z \to X[1] \right) \in \Delta$$

- $\forall X \in \mathcal{T}$
$$\left(X \xrightarrow{1_X} X \to 0 \to X[1] \right) \in \Delta$$

- Δ is closed under isomorphisms

(T2) For any $\left(X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]\right) \in \Delta$
otation) have

$$\left(Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]\right) \in \Delta$$

$$\& \quad \left(Z[-1] \xrightarrow{-h[-1]} X \xrightarrow{f} Y \xrightarrow{g} Z\right) \in \Delta$$

(T3) Given solid part
out of 3)

$$\begin{tikzcd}
(X \arrow[r] \arrow[d, "u"] & Y \arrow[r] \arrow[d, "v"'] & Z \arrow[r] \arrow[d, dashed, "w"] & X[1]) \in \Delta \arrow[d, "u[1]"] \\
(X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]) \in \Delta
\end{tikzcd}$$

where ↑ commutes, then can find morphism $w: Z \to Z'$ s.t. entire diagram commutes

(T4) Given solid part of diagram, where the
ctahedral
axiom) two rows & column are in Δ

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d] & Z' \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, equal] \\
X \arrow[r] & Z \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, "f[1]"] \\
 & X' \arrow[r, equal] \arrow[d, "h"'] & X' \arrow[r, "h"] \arrow[d] & Y[1] \\
 & Y[1] \arrow[r, "g[1]"] & Z'[1] &
\end{tikzcd}$$

there are dashed arrows as indicated making the whole diagram commutative, and s.t. third column is in Δ.

Remark

- By Δ closed under iso we mean

$$\begin{tikzcd}
\big(X \arrow[r] \arrow[d, "u"'] & Y \arrow[r] \arrow[d, "v"'] & Z \arrow[r] \arrow[d, "w"'] & X[1] \big) \in \Delta \arrow[d, "u[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

commutative & u, v, w iso

$\Rightarrow (X' \to Y' \to Z' \to X'[1]) \in \Delta$

- $[1]$ called the suspension, $[n] = \underbrace{[1] \circ \ldots \circ [1]}_{n}$

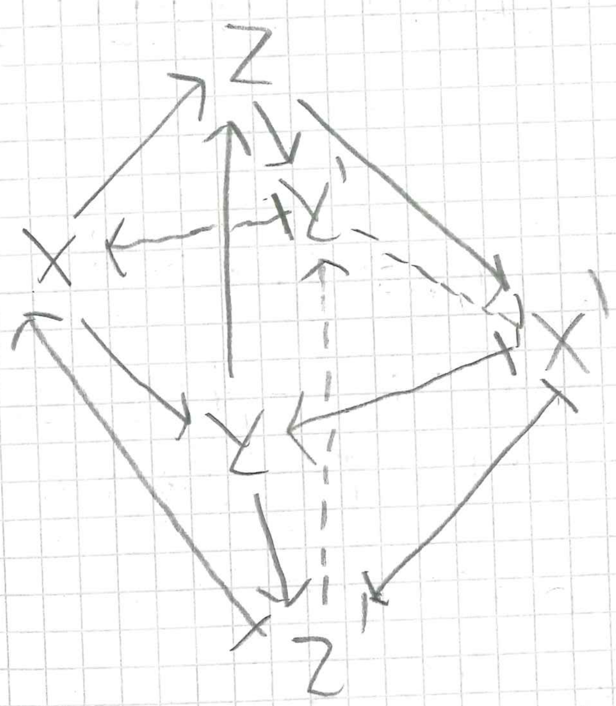
$[-1]$ a quasi-inverse of $[1]$, $[-n] = \underbrace{[-1] \circ \ldots \circ [-1]}_{n}$

- elements in Δ called distinguished triangles
- Triangulated categories were simultanously introduced by Verdier ('63) motivated by structures in algebra & algebraic geometry (Derived category) and by Puppe ('62) from structures in algebraic topolodgy (minus (T4))
- Open question whether (T4) follows from (T1)-(T3)

- Morphisms $Z \to X[1]$ sometimes denoted $Z \xrightarrow{+} X$. Then elts of Δ are triangles

$$\begin{tikzcd} X \arrow[rr] & & Y \arrow[dl] \\ & Z \arrow[ul, "+"] & \end{tikzcd}$$

and (T4) becomes an actual octahedron



All oriented triangles lie in Δ
All non-oriented triangles commute.

- $\mathcal{T}$ triangulated $\Rightarrow \mathcal{T}^{op}$ triangulated

<u>Theorem</u> (Long exact Hom-sequence)

$(\mathcal{J}, [1], \Delta)$ triang cat, $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \in \Delta$, $T \in \mathcal{J}$. Then

$$\cdots \xrightarrow{h[n]\circ -} \mathrm{Hom}_{\mathcal{J}}(T, X[n]) \xrightarrow{f[n]\circ -} \mathrm{Hom}_{\mathcal{J}}(T, Y[n]) \xrightarrow{g[n]\circ -} \mathrm{Hom}_{\mathcal{J}}(T, Z[n]) \longrightarrow \mathrm{Hom}_{\mathcal{J}}(T, X[n+1]) \xrightarrow{f[n+1]\circ -} \mathrm{Hom}_{\mathcal{J}}(T, Y[n+1]) \xrightarrow{g[n+1]\circ -} \mathrm{Hom}_{\mathcal{J}}(T, Z[n+1]) \to \cdots$$

&

$$\cdots \xrightarrow{-\circ h[n-1]} \mathrm{Hom}_{\mathcal{J}}(Z[n], T) \xrightarrow{-\circ g[n]} \mathrm{Hom}_{\mathcal{J}}(Y[n], T) \xrightarrow{-\circ f[n]} \mathrm{Hom}_{\mathcal{J}}(X[n], T) \longrightarrow \mathrm{Hom}_{\mathcal{J}}(Z[n-1], T) \xrightarrow{-\circ g[n-1]} \mathrm{Hom}_{\mathcal{J}}(Y[n-1], T) \xrightarrow{-\circ f[n-1]} \mathrm{Hom}_{\mathcal{J}}(X[n-1], T) \to \cdots$$

are exact.

<u>Pf</u>: Prove first claim, second dual.
By rotation axiom (T2), suffices to show

$$\mathrm{Hom}_{\mathcal{J}}(T, X) \xrightarrow{f\circ -} \mathrm{Hom}_{\mathcal{J}}(T, Y) \xrightarrow{g\circ -} \mathrm{Hom}_{\mathcal{J}}(T, Z)$$

exact. Consider

$$\begin{tikzcd}
T \arrow[r, "1"] \arrow[d, dashed, "u"] & T \arrow[r] \arrow[d, "v"] & 0 \arrow[r] \arrow[d, dashed] & T[1] \arrow[d, dashed, "u[1]"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & X[1]
\end{tikzcd}$$

by (T3) existence of u s.t. left square commute

$\Leftrightarrow \exists$ morphism $0 \to Z$ s.t.

$$\begin{array}{ccc} T & \longrightarrow & 0 \\ {\scriptstyle v}\downarrow & & \downarrow \\ X & \xrightarrow{g} & Y \end{array}$$

commutes

$\Leftrightarrow g \circ v = 0$. Summarizing

$g \circ v = 0 \Leftrightarrow \exists u \in Hom_{\mathcal{J}}(T, X)$ s.t. $v = f \circ u$.

This shows exactness. ∎

<u>Lemma</u>: $f: X \to Y$ morphism in cat $\mathcal{C}$. The following are equivalent:

- f is an isomorphism
- $Hom_{\mathcal{C}}(Z, X) \xrightarrow{f \circ -} Hom_{\mathcal{C}}(Z, Y)$ bijective $\forall Z \in \mathcal{C}$
- $Hom_{\mathcal{C}}(Y, Z) \xrightarrow{- \circ f} Hom_{\mathcal{C}}(X, Z)$ bijective $\forall Z \in \mathcal{C}$.

<u>Pf</u>: Exercise ∎

<u>Theorem</u>: (2 out of 3 property for isomorphisms)

$(\mathcal{J}, [1], \Delta)$ triang cat. Consider comm diagram

$$\begin{array}{ccccccc} X_1 & \longrightarrow & Y_1 & \longrightarrow & Z_1 & \longrightarrow & X_1[1] & \in \Delta \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] & \\ X_2 & \longrightarrow & Y_2 & \longrightarrow & Z_2 & \longrightarrow & X_2[1] & \in \Delta \end{array}$$