

- F is exact
- F is left exact and preserves epimorphism
- F is right exact and preserves monomorphisms

Remark: A contravariant functor F from \mathcal{A} to \mathcal{B} is said to be left or right exact if the associated functor $F: \mathcal{A}^{\text{op}} \rightarrow \mathcal{B}$

is left or right exact, respectively.

Note that $F: \mathcal{A}^{\text{op}} \rightarrow \mathcal{B}$ left (right) exact iff $F^{\text{op}}: \mathcal{A} \rightarrow \mathcal{B}$ is right (left) exact:

Recall: \mathcal{C} cat, $X \in \mathcal{C}$, have functors

$$\text{Hom}_{\mathcal{C}}(X, -): \mathcal{C} \rightarrow \text{Set}$$

$$\text{Hom}_{\mathcal{C}}(-, X): \mathcal{C}^{\text{op}} \rightarrow \text{Set}$$

If \mathcal{A} preadditive cat, $X \in \mathcal{A}$, then

$$\text{Hom}_{\mathcal{A}}(X, -): \mathcal{A} \rightarrow \text{Ab}$$

$$\text{Hom}_{\mathcal{A}}(-, X): \mathcal{A}^{\text{op}} \rightarrow \text{Ab}$$

Theorem (Hom is left exact)

\mathcal{A} abelian cat, $X \in \mathcal{A}$. Then the functors

$$\text{Hom}_{\mathcal{A}}(X, -): \mathcal{A} \rightarrow \text{Ab}$$

$$\text{Hom}_{\mathcal{A}}(-, X): \mathcal{A}^{\text{op}} \rightarrow \text{Ab}$$

are left exact.

Pf. $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ exact.

Want to show

$$0 \rightarrow \text{Hom}_{\mathcal{A}}(X, A) \xrightarrow{\text{Hom}_{\mathcal{A}}(X, f)} \text{Hom}_{\mathcal{A}}(X, B) \xrightarrow{\text{Hom}_{\mathcal{A}}(X, g)} \text{Hom}_{\mathcal{A}}(X, C)$$

is exact. This is a sequence in Ab , so can check on elements!

(i) $\text{Hom}_{\mathcal{A}}(X, f)$ injective.

Assume $h \in \text{Hom}_{\mathcal{A}}(X, A)$ satisfies

$$0 = \text{Hom}_{\mathcal{A}}(X, f)(h) = f \circ h$$

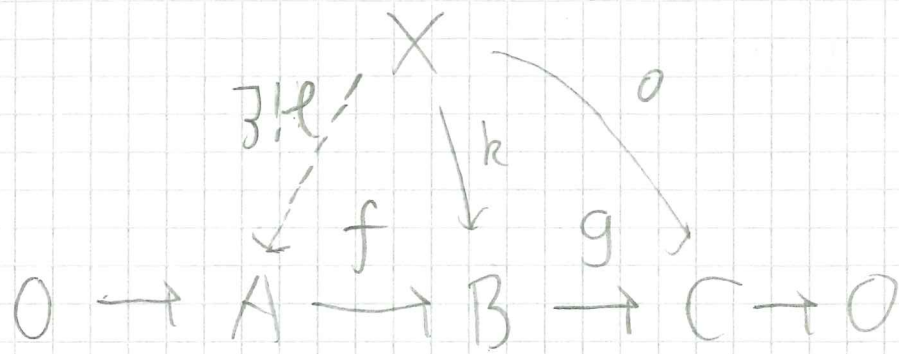
Since $f \circ h = 0 = f \circ 0$ and f mono

$$\Rightarrow h = 0, \quad \checkmark$$

(ii) Exact at $\text{Hom}_{\mathcal{A}}(X, B)$:

Let $k \in \ker \text{Hom}_{\mathcal{A}}(X, g)$, so

$$g \circ k = 0$$



Since f kernel of g , \exists unique map

$$l: X \rightarrow A \text{ s.t. } f \circ l = k, \text{ i.e.}$$

$$\text{Hom}_{\mathcal{A}}(X, f)(l) = k \quad \checkmark$$

This shows $\text{Hom}_{\mathcal{A}}(X, -): \mathcal{A} \rightarrow \text{Ab}$ is left exact. The fact that

$\text{Hom}_{\mathcal{A}}(-, X): \mathcal{A}^{\text{op}} \rightarrow \text{Ab}$ is left exact is proved dually.

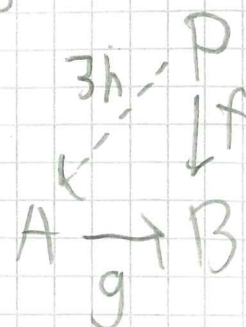
Q: When are $\text{Hom}_{\mathcal{A}}(X, -)$ & $\text{Hom}_{\mathcal{A}}(-, X)$ exact?

Def: \mathcal{A} abelian

An obj $P \in \mathcal{A}$ is projective if for all diagrams

with g epi, there exists $h: P \rightarrow A$

with $g \circ h = f$



- An obj $I \in \mathcal{A}$ is injective if for all diagrams

$$\begin{array}{ccc}
 & I & \\
 & \nwarrow \text{3h} & \\
 f \uparrow & & \\
 0 \rightarrow A & \xrightarrow{g} & B
 \end{array}$$

with g mono there exists $h: B \rightarrow I$
s.t. $h \circ g = f$

Remark: In the def of projective and injective the morphism h is not necessarily unique.

Proposition \mathcal{A} abelian. Then

- (1) $P \in \mathcal{A}$ is projective $\Leftrightarrow \text{Hom}_{\mathcal{A}}(P, -)$ exact
- (2) $I \in \mathcal{A}$ is injective $\Leftrightarrow \text{Hom}_{\mathcal{A}}(-, I)$ exact

Pf: Know $\text{Hom}_{\mathcal{A}}(P, -): \mathcal{A} \rightarrow \text{Ab}$ left exact.

So $\text{Hom}_{\mathcal{A}}(P, -)$ exact

$\Leftrightarrow \text{Hom}_{\mathcal{A}}(P, -)$ preserves epimorphisms

$\Leftrightarrow \forall$ epis $g: A \twoheadrightarrow B$, the map

$\text{Hom}_{\mathcal{A}}(P, g): \text{Hom}_{\mathcal{A}}(P, A) \rightarrow \text{Hom}_{\mathcal{A}}(P, B)$ ep

$\Leftrightarrow \forall$ epis $g: A \rightarrow B$ & \forall morphisms $f: P \rightarrow B$,
there exists $h: P \rightarrow A$ s.t.
 $f = \text{Hom}_{\mathcal{A}}(P, g)(h) = g \circ h$

$\Leftrightarrow P$ projective.
Similarly for I

Note:

- (i) $0 \in \mathcal{A}$ projective
- (ii) $P, Q \in \mathcal{A}$ projective $\Rightarrow P \oplus Q$ projective
- (iii) $\{P_i\}_{i \in I}$ set of projective objects.
If $\coprod_{i \in I} P_i$ exists, then it is projective.
- (iv) If P projective and $P \cong P_1 \oplus P_2$, then
 P_1 & P_2 are projective.

Now assume $\mathcal{A} = \text{Mod } R$, Ring. Then

- (v) $R \in \text{Mod } R$ projective since
 $\text{Hom}_R(R, -): \text{Mod } R \rightarrow \mathcal{A}$
is just the forgetful functor.

(vi) I set. Recall that

$$R^{(I)} = \left\{ \text{maps } f: I \rightarrow R \mid f(x) \neq 0 \text{ for only finitely many } x \in I \right\}$$

$$\text{Then } R^{(I)} \cong \bigoplus_{i \in I} R = \bigsqcup_{i \in I} R$$

Hence by (iii) & (v) $R^{(I)}$ is a projective R -module.

vii) If F free R -module with basis $\{v_i\}_{i \in I}$, then $F \cong \bigoplus_{i \in I} R$, so any free module is projective.

viii) If M summand of a free R -module, then M is projective by (iv) & (vii).

In fact, the converse of (viii) holds:

Theorem R ring, $P \in \text{Mod } R$.

The following are equivalent:

(i) P is projective

(ii) P is a direct summand of a free module

Proof: Have seen (ii) \Rightarrow (i), so only show (i) \Rightarrow (ii)

Let $\{x_i\}_{i \in I}$ be a generating set of P .

Then have epimorphism $\bigoplus_{i \in I} R \rightarrow P$

$$e_i \mapsto p(e_i) = x_i$$

Since p is epi, can find filling of the diagram

$$\begin{array}{ccc} & \exists i: & P \\ & \swarrow & \downarrow \text{id}_P \\ \bigoplus_{i \in I} R & \xrightarrow{p} & P \rightarrow 0 \end{array}$$

Then $po_i = \text{id}_P$ so P is a direct summand of $\bigoplus_{i \in I} R$ (have $\bigoplus_{i \in I} R \cong \ker p \oplus P$, check this!)

Remark: From the proof we see that
 $\forall M \in \text{Mod } R \quad \exists P \in \text{Mod } R$ projective
and epi $P \rightarrow M$

Can show dually that exists mono $M \rightarrow I$
with I injective, but much more difficult!

Tensor products

Motivation: Let ${}_R N_S$ be an R - S bimodule.
(left R -module & right S -module s.t.)
 $(rn)s = r(ns) \quad \forall r \in R, n \in N, s \in S$

Then $\forall M \in \text{Mod } R$, $\text{Hom}_R(N, M)$ is
a left S -module via $f \in \text{Hom}_R(N, M)$

$$(s \cdot f)(m) = f(ms)$$

→ Get Hom-functor

$$\text{Hom}_R(N, -) : \text{Mod } R \rightarrow \text{Mod } S$$

It turns out this has a left adjoint!

Our goal: Construct this left adjoint.

Def R ring, M right R -module, N left R -module

$F_{(M,N)}$ - free abelian group with basis $M \times N$, i.e.

$$F_{(M,N)} = \left\{ \sum_{\text{finite sum}} a_i (m_i, n_i) \mid a_i \in \mathbb{Z}, m_i \in M, n_i \in N \right\}$$

The tensor product of M and N , denoted $M \otimes_R N$, is the abelian group defined as follow

$$M \otimes_R N = \frac{F_{(M,N)}}{\left\{ \begin{array}{l} (m, n_1 + n_2) - (m, n_1) - (m, n_2), \forall m \in M, n_1, n_2 \in N \\ (m_1 + m_2, n) - (m_1, n) - (m_2, n), \forall m_1, m_2 \in M, n \in N \\ (m, r \cdot n) - (m \cdot r, n), \forall m \in M, n \in N, r \in R \end{array} \right.}$$

Denote the image of basis elt $(m, n) \in F$ in $M \otimes_R N$ by $m \otimes n$, called elementary tensor

Note: • elements in $M \otimes_R N$ are finite sums $\sum m_i \otimes n_i$, and may not be given by just