- F is exact
- F is left exact and preserves epimorphism
- F is right exact and preserves monomorphisms

Remark: A contravariant functor F from $\mathcal{A}$ to $\mathcal{B}$ is said to be left or right exact if the associated functor

$$F: \mathcal{A}^{op} \to \mathcal{B}$$

is left or right exact, respectively.
Note that $F: \mathcal{A}^{op} \to \mathcal{B}$ left (right) exact iff $F^{op}: \mathcal{A} \to \mathcal{B}$ is right (left) exact.

Recall: $\mathcal{C}$ cat, $X \in \mathcal{C}$, have functors

$$Hom_{\mathcal{C}}(X, -): \mathcal{C} \to Set$$

$$Hom_{\mathcal{C}}(-, X): \mathcal{C}^{op} \to Set.$$

If $\mathcal{A}$ preadditive cat, $X \in \mathcal{A}$, then

$$Hom_{\mathcal{A}}(X, -): \mathcal{A} \to Ab$$

$$Hom_{\mathcal{A}}(-, X): \mathcal{A}^{op} \to Ab.$$

<u>Theorem</u> (Hom is left exact)

$\mathcal{A}$ abelian cat, $X \in \mathcal{A}$. Then the functors

$$\mathrm{Hom}_{\mathcal{A}}(X,-) : \mathcal{A} \to Ab$$
$$\mathrm{Hom}_{\mathcal{A}}(-,X) : \mathcal{A}^{op} \to Ab$$

are left exact.

<u>Pf</u>: $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ exact.

Want to show

$$0 \to \mathrm{Hom}_{\mathcal{A}}(X,A) \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(X,f)} \mathrm{Hom}_{\mathcal{A}}(X,B) \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(X,g)} \mathrm{Hom}_{\mathcal{A}}(X,C)$$

is exact. This is a sequence in Ab, so can check on elements!

(i) $\mathrm{Hom}_{\mathcal{A}}(X,f)$ injective.

Assume $h \in \mathrm{Hom}_{\mathcal{A}}(X,A)$ satisfies

$0 = \mathrm{Hom}_{\mathcal{A}}(X,f)(h) = f \circ h$

Since $f \circ h = 0 = f \circ 0$ and f mono

$\Rightarrow h = 0$ ✓

(ii) Exact at $\mathrm{Hom}_{\mathcal{A}}(X,B)$:

Let $k \in \ker \mathrm{Hom}_{\mathcal{A}}(X,g)$ so

$g \circ k = 0$

$$\begin{tikzcd}
 & & X \arrow[dll, dashed, "\exists! \ell"'] \arrow[d, "k"] \arrow[dr, bend left, "0"] & & \\
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

Since f kernel of g, $\exists$ unique map $\ell : X \to A$ s.t. $f \circ \ell = k$, i.e. $\mathrm{Hom}_{\mathcal{A}}(X, f)(\ell) = k$. ✓

This shows $\mathrm{Hom}_{\mathcal{A}}(X, -) : \mathcal{A} \to \mathrm{Ab}$ is left exact. The fact that $\mathrm{Hom}_{\mathcal{A}}(-, X) : \mathcal{A}^{op} \to \mathrm{Ab}$ is left exact is proved dually ∎

Q: When are $\mathrm{Hom}_{\mathcal{A}}(X, -)$ & $\mathrm{Hom}_{\mathcal{A}}(-, X)$ exact?

<u>Def</u>: $\mathcal{A}$ abelian

- An obj $P \in \mathcal{A}$ is <u>projective</u> if for all diagrams

$$\begin{tikzcd}
 & & P \arrow[dl, dashed, "\exists h"'] \arrow[d, "f"] & \\
 0 \arrow[r] & A \arrow[r, "g"'] & B \arrow[r] & 0
\end{tikzcd}$$

with g epi there exists $h : P \to A$ with $g \circ h = f$

- An obj $I \in \mathcal{A}$ is injective if for all diagrams

$$\begin{tikzcd} & I & \\ 0 \arrow[r] & A \arrow[u, "f"] \arrow[r, "g"] & B \arrow[ul, dashed, "\exists h"'] \end{tikzcd}$$

with g mono there exists $h: B \to I$ s.t. $h \circ g = f$

Remark: In the def of projective and injective the morphism h is not necessarily unique.

Proposition $\mathcal{A}$ abelian. Then

(1) $P \in \mathcal{A}$ is projective $\Leftrightarrow \mathrm{Hom}_{\mathcal{A}}(P,-)$ exact

(2) $I \in \mathcal{A}$ is injective $\Leftrightarrow \mathrm{Hom}_{\mathcal{A}}(-,I)$ exact

Pf: Know $\mathrm{Hom}_{\mathcal{A}}(P,-): \mathcal{A} \to Ab$ left exact

So $\mathrm{Hom}_{\mathcal{A}}(P,-)$ exact

$\Leftrightarrow \mathrm{Hom}_{\mathcal{A}}(P,-)$ preserves epimorphisms

$\Leftrightarrow \forall$ epis $g: A \twoheadrightarrow B$, the map

$\mathrm{Hom}_{\mathcal{A}}(P,g): \mathrm{Hom}_{\mathcal{A}}(P,A) \to \mathrm{Hom}_{\mathcal{A}}(P,B)$ ep

$\Leftrightarrow$ $\forall$ epis $g: A \twoheadrightarrow B$ & $\forall$ morphisms $f: P \to B$, there exists $h: P \to A$ s.t.
$f = \mathrm{Hom}_{\mathcal{A}}(P, g)(h) = g \circ h$

$\Leftrightarrow$ P projective.

Similarly for I ∎

Note:

(i) $0 \in \mathcal{A}$ projective

(ii) $P, Q \in \mathcal{A}$ projective $\Rightarrow P \oplus Q$ projective

(iii) $\{P_i\}_{i \in I}$ set of projective objects. If $\coprod_{i \in I} P_i$ exists, then it is projective.

(iv) If P projective and $P \cong P_1 \oplus P_2$, then P_1 & P_2 are projective.

Now assume $\mathcal{A} = \mathrm{Mod}\, R$, R ring. Then

(v) $R \in \mathrm{Mod}\, R$ projective since
$\mathrm{Hom}_R(R, -): \mathrm{Mod}\, R \to \mathrm{Ab}$
is just the forgetful functor.

(vi) I set. Recall that

$$R^{(I)} = \{\text{maps } f: I \to R \mid f(x) \neq 0 \text{ for only finitely many } x \in I\}$$

Then $R^{(I)} \cong \bigoplus_{i \in I} R = \coprod_{i \in I} R$

Hence by (iii) & (v) $R^{(I)}$ is a projective R-module.

(vii) If F free R-module with basis $\{v_i\}_{i \in I}$, then $F \cong \bigoplus_{i \in I} R$, so any free module is projective

(viii) If M summand of a free R-module, then M is projective by (iv) & (vii)

In fact, the converse of (viii) holds:

<u>Theorem</u> R ring, $P \in \operatorname{Mod} R$.
The following are equivalent:

(i) P is projective

(ii) P is a direct summand of a free module

<u>Proof</u>: Have seen (ii) $\Rightarrow$ (i), so only show (i) $\Rightarrow$ (ii)

Let $\{x_i\}_{i \in I}$ be a generating set of P.

Then have epimorphism $\bigoplus_{i \in I} R \xrightarrow{p} P$

$$e_i \longmapsto p(e_i) = x_i$$

Since p is epi, can find filling of the diagram

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists i"'] \arrow[d, "id_P"] & \\ \bigoplus_{i \in I} R \arrow[r, "p"] & P \arrow[r] & 0 \end{tikzcd}$$

Then $p \circ i = id_P$ so P is a direct summand of $\bigoplus_{i \in I} R$ (have $\bigoplus_{i \in I} R \cong \ker p \oplus P$, check this!)

Remark: From the proof we see that $\forall M \in \text{Mod}\, R$ $\exists P \in \text{Mod}\, R$ projective and epi $P \twoheadrightarrow M$

Can show dually that exists mono $M \rightarrowtail I$ with I injective, but much more difficult!

Tensor products

Motivation: Let ${}_R N_S$ be an R-S bimodule.
(left R-module & right S-module s.t.
$(rn)s = r(ns)$ $\forall r \in R, n \in N, s \in S$)

Then $\forall M \in \text{Mod}\, R$, $\text{Hom}_R(N, M)$ is a left S-module via $f \in \text{Hom}_R(N, M)$

$$(s \cdot f)(m) = f(ms).$$

$\rightarrow$ Get Hom-functor

$$\text{Hom}_R(N, -) : \text{Mod}\, R \rightarrow \text{Mod}\, S$$

It turns out this has a left adjoint!

Our goal: Construct this left adjoint.

<u>Def</u> R ring, M right R-module, N left R-module

$F_{(M,N)}$ - free abelian group with basis $M\times N$, i.e.

$$F_{(M,N)} = \Big\{ \sum_{\substack{\text{finite}\\ \text{sum}}} a_i (m_i, n_i) \mid a_i \in \mathbb{Z},\ m_i \in M,\ n_i \in N \Big\}$$

The <u>tensor product</u> of M and N, denoted $M \otimes_R N$, is the abelian group defined as follow

$$M \otimes_R N = \frac{F_{(M,N)}}{\left\langle \begin{array}{l} (m, n_1+n_2) - (m, n_1) - (m, n_2),\ \forall m \in M,\ n_1, n_2 \in N \\ (m_1+m_2, n) - (m_1, n) - (m_2, n),\ \forall m_1, m_2 \in M,\ n \in N \\ (m, r\cdot n) - (m\cdot r, n),\ \forall m \in M,\ n \in N,\ r \in R \end{array} \right\rangle}$$

Denote the image of basis elt $(m,n) \in F$ in $M \otimes_R N$ by $m \otimes n$, called <u>elementary tensor</u>

Note: • elements in $M \otimes_R N$ are finite sums $\sum m_i \otimes n_i$, and may not be given by just