Furthermore, $\mathrm{Cone}(-f^\bullet) = \mathrm{Tot}(X^{\bullet,\bullet})$

From

$$X^{\bullet,a}[a-1] \xrightarrow{f^\bullet} \mathrm{Tot}(Y^{\bullet,\bullet}) \to \mathrm{Cone}(-f^\bullet) \to X^{\bullet,a}[-a]$$

we get a long exact sequence in homology.

Hence, since $X^{\bullet,a}$ is exact by assumption, we see that $\mathrm{Tot}(X^{\bullet,\bullet}) = \mathrm{Cone}(-f^\bullet)$ is exact iff $\mathrm{Tot}(Y^{\bullet,\bullet})$ is exact

Now since $Y^{\bullet,\bullet}$ has less non-zero rows than $X^{\bullet,\bullet}$, we may assume by induction that $\mathrm{Tot}(Y^{\bullet,\bullet})$ is exact. This proves the claim ▪

Lecture 18

**Corollary**: $X^{\bullet,\bullet}$ double complex, such that each diagonal has only finitely many $\neq 0$ objects. Assume all rows of $X^{\bullet,\bullet}$ are exact as complexes. Then $\mathrm{Tot}(X^{\bullet,\bullet})$ is exact.

**Proof**: To check exactness at a given position $s$ in $\mathrm{Tot}(X^{\bullet,\bullet})$, may disregard $n$ s.t. $X^{n,s-n} = 0$. Hence exactness follows from the previous proposition ▪

Theorem (Balancing Ext)

$\mathcal{A}$ abelian cat with enough proj and inj. Then for any $A, B \in \mathcal{A}$

$$\mathrm{Ext}^n_{\mathcal{A}}(A,-)(B) \cong \mathrm{Ext}^n_{\mathcal{A}}(-,B)(A)$$

Proof: Write

$$\left.\begin{array}{l} P^\bullet = (\cdots \to P^{-2} \to P^{-1} \to P^0 \to 0 \to \cdots) \\ I^\bullet = (\cdots \to 0 \to I^0 \to I^1 \to I^2 \to \cdots) \end{array}\right\}$$

for proj resolution of $A$ and inj coresolution of $B$, respectively.

$\mathrm{Ext}^n_{\mathcal{A}}(A,-)(B)$ - homology of

$$\mathrm{Hom}_{\mathcal{A}}(A, I^\bullet) = \cdots \to 0 \to \mathrm{Hom}_{\mathcal{A}}(A, I^0) \to \mathrm{Hom}_{\mathcal{A}}(A, I^1) \to \cdots$$

$\mathrm{Ext}^n_{\mathcal{A}}(-,B)(A)$ - homology of

$$\mathrm{Hom}_{\mathcal{A}}(P^\bullet, B) = (\cdots \to 0 \to \mathrm{Hom}_{\mathcal{A}}(P^0, B) \to \mathrm{Hom}_{\mathcal{A}}(P^1, B) \to \cdots)$$

Suffices to show $\mathrm{Hom}_{\mathcal{A}}(A, I^\bullet)$ and $\mathrm{Hom}_{\mathcal{A}}(P^\bullet, B)$ can be connected via quasi-isomorphisms

Strategy: Show there exists quasi-iso's

$$\mathrm{Hom}_{\mathcal{A}}(P^\bullet, B) \longrightarrow \mathrm{Tot}(\mathrm{Hom}(P^\bullet, I^\bullet)) \longleftarrow \mathrm{Hom}_{\mathcal{A}}(A, I^\bullet)$$

where $\mathrm{Hom}_{\mathcal{A}}(P^\bullet, I^\bullet)^{ij} = \mathrm{Hom}_{\mathcal{A}}(P^{-i}, I^j)$

as a double complex

Write

$$\bar{P}^\bullet = (\cdots \to P^{-1} \to P^0 \to A \to 0 \to \cdots)$$

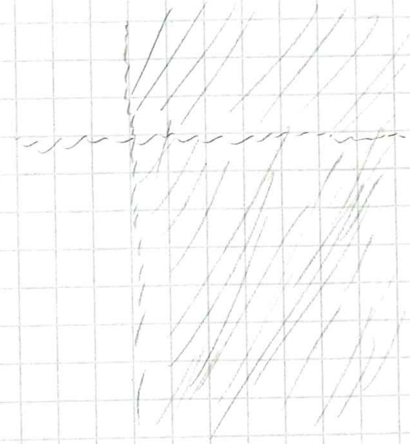
$$\bar{I}^\bullet = (\cdots \to 0 \to B \to I^0 \to I^1 \to \cdots)$$

$\bar{P}^\bullet, \bar{I}^\bullet$ exact complexes. In the following we write $(X, Y)$ for $\mathrm{Hom}_{\mathcal{A}}(X, Y)$

$$\begin{tikzcd}
(A, B) \arrow[r] \arrow[d] & (P^0, B) \arrow[r] \arrow[d] & (P^{-1}, B) \arrow[r] \arrow[d] & (P^{-2}, B) \arrow[r] \arrow[d] & \cdots \\
(A, I^0) \arrow[r] \arrow[d] & (P^0, I^0) \arrow[r] \arrow[d] & (P^{-1}, I^0) \arrow[r] \arrow[d] & (P^{-2}, I^0) \arrow[r] \arrow[d] & \cdots \\
(A, I^1) \arrow[r] \arrow[d] & (P^0, I^1) \arrow[r] \arrow[d] & (P^{-1}, I^1) \arrow[r] \arrow[d] & (P^{-2}, I^1) \arrow[r] \arrow[d] & \cdots \\
(A, I^2) \arrow[r] \arrow[d] & (P^0, I^2) \arrow[r] \arrow[d] & (P^{-1}, I^2) \arrow[r] \arrow[d] & (P^{-2}, I^2) \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

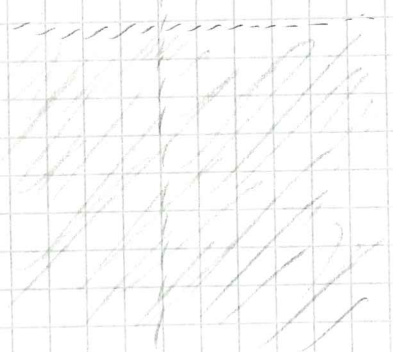
Note

$\mathrm{Hom}_{\mathcal{A}}(P^{\bullet}, \bar{I}^{\bullet}) =$



and

$\mathrm{Hom}_{\mathcal{A}}(\bar{P}^{\bullet}, I^{\bullet}) =$



as double complexes. Since $\mathrm{Hom}_{\mathcal{A}}(P^{\bullet}, \bar{I}^{\bullet})$ has exact columns and $\mathrm{Hom}_{\mathcal{A}}(\bar{P}^{\bullet}, I^{\bullet})$ has exact rows $\Rightarrow$ $\mathrm{Tot}(\mathrm{Hom}_{\mathcal{A}}(P^{\bullet}, \bar{I}^{\bullet}))$ & $\mathrm{Tot}(\mathrm{Hom}_{\mathcal{A}}(\bar{P}^{\bullet}, I^{\bullet}))$ exact.

Next, note that we have morphism

$\mathrm{Hom}_{\mathcal{A}}(A, I^{\bullet}) \longrightarrow \mathrm{Tot}\,\mathrm{Hom}_{\mathcal{A}}(P^{\bullet}, I^{\bullet})$

$\mathrm{Hom}_{\mathcal{A}}(P^{\bullet}, B) \longrightarrow \mathrm{Tot}\,\mathrm{Hom}_{\mathcal{A}}(P^{\bullet}, I^{\bullet})$

given by the morphisms $\rightarrow$ and $\downarrow$ below, respectively


Their cones are $\mathrm{Tot}(\mathrm{Hom}_{\mathcal{A}}(\overline{P}^\cdot, I^\cdot))$ and $\mathrm{Tot}(\mathrm{Hom}_{\mathcal{A}}(P^\cdot, \overline{I}^\cdot))$, which are exact. Hence the maps are quasi-isomorphisms.

So we have quasi-iso's

$$\mathrm{Hom}_{\mathcal{A}}(P^\cdot, B) \xrightarrow{\simeq} \mathrm{Tot}(\mathrm{Hom}_{\mathcal{A}}(P^\cdot, I^\cdot)) \xleftarrow{\simeq} \mathrm{Hom}_{\mathcal{A}}(A, I^\cdot)$$

Taking nth homology, get iso's

$$\mathrm{Ext}^n_{\mathcal{A}}(-, B)(A) \xrightarrow{\cong} H^n\,\mathrm{Tot}(\mathrm{Hom}_{\mathcal{A}}(P^\cdot, I^\cdot)) \xleftarrow{\cong} \mathrm{Ext}^n_{\mathcal{A}}(A, -)(B)$$

■

<u>Theorem</u>: $R$ ring. Then for any right $R$-module $M$ and left $R$-module $N$ we have

$$\mathrm{Tor}_i^R(M, -)(N) \cong \mathrm{Tor}_i^R(-, N)(M)$$

<u>Proof</u>: Exercise ■

# Homological dimensions

Definition $\mathcal{A}$ abelian cat, $A \in \mathcal{A}$

(1) Assume $\mathcal{A}$ has enough projectives.
Then the projective dimension of $A$, denoted $\text{pd}\, A$, is the smallest $n \geq 0$ s.t. exists projective resolution $0 \to P^{-n} \to \dots \to P^{-1} \to P^0 \to 0$
(write $\text{pd}\, A = \infty$ if all proj resolutions are infinite)

(2) Assume $\mathcal{A}$ has enough injectives.
Then the injective dimension of $A$, denoted $\text{id} A$, is the smallest $n \geq 0$ s.t. exist injective coresolution
$0 \to I^0 \to I^1 \to \dots \to I^n \to 0$.
(write $\text{id}\, A = \infty$ if all injective coresolution are infinite)

Note: $\text{pd}\, A$ and $\text{id}\, A$ can be interpreted as a measure of how far $A$ is from being projective and injective. In particular,

$\text{pd}\, A = 0 \iff A$ proj

$\text{id}\, A = 0 \iff A$ inj

Our goal is to show some equivalent characterizations of projective and injective dimension. For this, need the following lemma.

<u>Lemma</u>: $\mathcal{A}$ abelian, $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ exact in $\mathcal{A}$. The following are equivalent:

(1) $i$ is a split monomorphism

(2) $p$ is a split epimorphism

(3) There exists an isomorphism $\phi: B \xrightarrow{\cong} A \oplus C$ s.t.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "="] & B \arrow[r, "p"] \arrow[d, "\phi"] & C \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & A \arrow[r, "\binom{1}{0}"] & A \oplus C \arrow[r, "{(0,1)}"] & C \arrow[r] & 0
\end{tikzcd}$$

is commutative

<u>Proof</u> Clearly $(3) \Rightarrow (2)$ & $(3) \Rightarrow (1)$

We show $(1) \Rightarrow (3)$. $(2) \Rightarrow (3)$ is shown dually

Assume $q: B \to A$ satisfies $q \circ i = id_A$

Define $\phi := (q, p): B \to A \oplus C$. Then have commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "="] & B \arrow[r] \arrow[d, "\phi"] & C \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & A \arrow[r, "\binom{1}{0}"] & A \oplus C \arrow[r, "(0,1)"] & C \arrow[r] & 0
\end{tikzcd}$$

By the five lemma $\phi$ is an isomorphism.
This proves the claim ∎

## Lecture 19

<u>Theorem</u>: $\mathcal{A}$ abelian cat with enough proj,
$A \in \mathcal{A}$, and $n \geq 0$ integer. The following are equivalent:

(1) $pd(A) \leq n$

(2) If $0 \to X \rightarrowtail P^{-(n-1)} \to P^{-(n-2)} \to \dots \to P^{-1} \to P^{0} \to A \to 0$ exact with $P^{-i}$ proj $\forall\, 0 \leq i \leq n-1$, then $X$ is projective

(3) $\mathrm{Ext}^{n+1}_{\mathcal{A}}(A, A') = 0 \quad \forall A' \in \mathcal{A}$

(4) $\mathrm{Ext}^{i}_{\mathcal{A}}(A, A') = 0 \quad \forall A' \in \mathcal{A} \quad \forall\, i \geq n+1$

<u>Proof</u>: Clearly (4) $\Rightarrow$ (3) and (2) $\Rightarrow$ (1)

Assume (1) and let $P^{\cdot}$ be a proj resolution of length $n$. Then $\mathrm{Ext}^i(A, A') \cong H^i(\mathrm{Hom}_{\mathcal{A}}(P^{\cdot}, A'))$
But $\mathrm{Hom}_{\mathcal{A}}(P^{\cdot}, A')$ must be concentrated