$$\begin{tikzcd}
 & & P^n \arrow[r, "d_P^n"] \arrow[d, "f^n"] \arrow[lld, dashed, "h^n"'] \arrow[ld, dashed] & P^{n+1} \arrow[d, "f^{n+1}"] \arrow[ld, "h^{n+1}"'] \\
E^{n-1} \arrow[rr, "d_E^{n-1}"] \arrow[rd, two heads] & & E^n \arrow[r, "d_E^n"'] & E^{n+1} \\
 & \operatorname{Im} d_E^{n-1} \arrow[ru, tail] & &
\end{tikzcd}$$

$$d_E^n \circ (f^n - h^{n+1} \circ d_P^n) = 0$$

$\Rightarrow f^n - h^{n+1} \circ d_P^n$ factors through $\operatorname{Im} d_E^{n-1} \rightarrowtail E^n$

P^n projective $\Rightarrow P^n \to \operatorname{Im} d_E^{n-1}$ lift along epi $E^{n-1} \twoheadrightarrow \operatorname{Im} d_E^{n-1}$ to a morphism $h^n : P^n \longrightarrow E^{n-1}$. Then $d_E^{n-1} \circ h^n = f^n - h^{n+1} \circ d_P^n$

$\Leftrightarrow f^n = d_E^{n-1} \circ h^n + h^{n+1} \circ d_P^n$

(2) $s^\bullet$ quasi-iso $\Leftrightarrow$ cone$(s^\bullet)$ exact.
Apply $\operatorname{Hom}_{K(\mathcal{C})}(P^\bullet, -)$ to triangle

$$X^\bullet \xrightarrow{s^\bullet} Y^\bullet \longrightarrow \operatorname{Cone}(s^\bullet) \to X^\bullet[1]$$

get exact seq

$$\ldots (P^\cdot, \mathrm{cone}(s^\cdot)[-1]) \to (P^\cdot, X) \to (P^\cdot, Y) \to (P^\cdot, \mathrm{cone}(s))$$
$$\overset{\|}{0} \text{ by (1)} \qquad\qquad \overset{\|}{0} \text{ by (1)}$$

$$\Rightarrow (P^\cdot, X) \to (P^\cdot, Y) \text{ iso.}$$

(3) $Q : K(\mathcal{A}) \to D(\mathcal{A})$ localization functor

The morphisms

$\mathrm{Hom}_{K(\mathcal{A})}(P^\cdot, X^\cdot) \to \mathrm{Hom}_{D(\mathcal{A})}(P^\cdot, X^\cdot)$ $\forall$ complexes $X^\cdot$

induces a natural transformation

$\phi_{P^\cdot} : \mathrm{Hom}_{K(\mathcal{A})}(P^\cdot, -) \to \mathrm{Hom}_{D(\mathcal{A})}(P^\cdot, Q(-))$

Since $\mathrm{Hom}_{K(\mathcal{A})}(P^\cdot, -) : K(\mathcal{A}) \to Ab$

sends quasi-iso's to iso's, it induces a

functor $\overline{\mathrm{Hom}_{K(\mathcal{A})}(P^\cdot, -)} : D(\mathcal{A}) \to Ab$

s.t. $\mathrm{Hom}_{K(\mathcal{A})}(P^\cdot, -) = \overline{\mathrm{Hom}_{K(\mathcal{A})}(P^\cdot, -)} \circ Q$

$id_{P^\cdot} \in \mathrm{Hom}_{K(\mathcal{A})}(P^\cdot, P^\cdot) = \overline{\mathrm{Hom}_{K(\mathcal{A})}(P^\cdot, -)}(Q(P^\cdot))$

Yoneda's lemma applied to $(\overline{\mathrm{Hom}_{K(\mathcal{A})}(P^\cdot, -)}, id_{P^\cdot})$

gives natural transformation

$$\mathrm{Hom}_{D(\mathcal{A})}(P^\bullet, -) \to \overline{\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, -)}$$

Precomposing with $Q^\bullet$ gives nat transf

$$\mathrm{Hom}_{D(\mathcal{A})}(P^\bullet, Q(-)) \longrightarrow \mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, -).$$

This is an inverse to $\phi_{P^\bullet}$. (check!) ∎

<u>Corollary</u>: Assume $\mathcal{A}$ has enough proj or inj. Then

$$\mathrm{Hom}_{D(\mathcal{A})}(A, B[n]) \cong \mathrm{Ext}^n_{\mathcal{A}}(A, B)$$

$\forall A, B \in \mathcal{A}$ $\forall n \geq 0$.

<u>Pf</u>: Assume $\mathcal{A}$ has enough proj. Enough inj dual.

pA = proj res of A.

$$\mathrm{Ext}^n_{\mathcal{A}}(A, B) \cong \mathrm{Hom}_{K(\mathcal{A})}(pA, B[n])$$

$$\xrightarrow{\text{prev Thm}} \cong \mathrm{Hom}_{D(A)}(pA, B[n])$$

$$\cong \mathrm{Hom}_{D(A)}(A, B[n])$$

$pA \to A$ quasi-iso, ∎