have comm diagram

$$\begin{tikzcd} \mathrm{Im} f \arrow[r, tail] \arrow[d, "k"] & B \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circ"] & \mathrm{Coker} f \arrow[d, dashed, "\exists ! l"] \\ \mathrm{Ker} g \arrow[r, tail] & B \arrow[r, two heads] & \mathrm{Im} g \end{tikzcd}$$

l exists by universal property of $\mathrm{Coker} f$.

Can show: k isomorphism $\Leftrightarrow$ l isomorphism.

<u>Def</u>: • A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ with $g \circ f = 0$ is <u>exact</u> if $k : \mathrm{Im} f \to \mathrm{Ker} g$ is an iso

• A sequence $(\cdots \xrightarrow{f_{n-2}} A_{n-1} \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \xrightarrow{f_{n+1}} \cdots)$ is <u>exact</u> if $A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1}$ is exact $\forall i \in \mathbb{Z}$

<u>Lecture 8</u>

<u>Rmk</u>: By abuse of notation sometimes say $\mathrm{Im} f = \mathrm{Ker} g$ (even though they are only iso)

(2) (i) $0 \to A \xrightarrow{f} B$ exact $\Leftrightarrow$ $\mathrm{Ker} f \cong 0$
$\Leftrightarrow$ f mono

(ii) $B \xrightarrow{g} C \to 0$ exact

$\iff \operatorname{Coker} g \cong 0 \iff g$ epimorphism

(iii) $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ exact

$\iff f$ mono & $\operatorname{Im} f = \operatorname{Ker} g \iff A \xrightarrow{f} B$ kernel of g

(iv) $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ exact

$\iff g$ epi & $\operatorname{Im} f = \operatorname{Ker} g \iff B \xrightarrow{g} C$ cokernel of f

(v) $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ exact

$\iff f$ kernel of g & g cokernel of f.

exact sequence as in (v) called
<u>short exact sequence</u>

<u>Lemma 2.1</u> $\mathcal{A}$ abelian,
commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "v"] \\ Z \arrow[r, "u"] & W \end{tikzcd} \quad (*)$$

in $\mathcal{A}$.

Then

(1) $(*)$ is a pullback

$\iff 0 \to X \xrightarrow{\binom{f}{g}} Y \oplus Z \xrightarrow{(v, -u)} W$ is exact

(2) (*) is a pushout

$\iff X \xrightarrow{\binom{f}{g}} Y \oplus Z \xrightarrow{(v,-u)} W \to 0$ is exact

<u>Pf</u>: (2) dual of (1)

For (1):

- A morphism $U \xrightarrow{\binom{a}{b}} Y \oplus Z$ satisfying $(v,-u)\binom{a}{b} = 0$

$\iff$ commutative square

$$\begin{tikzcd} U \arrow[r, "a"] \arrow[d, "b"'] & Y \arrow[d, "v"] \\ Z \arrow[r, "u"] & W \end{tikzcd}$$

- Morphism $U \xrightarrow{c} X$ satisfying $\binom{f}{g} \circ c = \binom{a}{b}$

$\iff$ commutative diagram

$$\begin{tikzcd}
U \arrow[dr, dashed, "c"] \arrow[drr, bend left, "a"] \arrow[ddr, bend right, "b"'] & & \\
& X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "v"] \\
& Z \arrow[r, "u"] & W
\end{tikzcd}$$

Hence the result follows from the universal property of a pullback and of a kernel ∎

<u>Corollary 2.2</u>: $\mathcal{A}$ abelian. If the square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "v"] \\
Z \arrow[r, "u"] & W
\end{tikzcd}$$

- Is a pullback, and u is epi, then it is also a pushout
- Is a pushout and f is mono, then it is also a pullback.

<u>Pf</u>: Follows from previous lemma and the fact that f is a mono $\Rightarrow \binom{f}{g}$ is a mono, and

u is epi $\Rightarrow (v,-u)$ is epi.

Corollary: Pullbacks and pushouts exist in abelian categories

Pf: From lemma we know that a pullback of $\begin{array}{c} Y \\ \downarrow v \\ Z \xrightarrow{u} W \end{array}$ is the same as a kernel of

$$Y \oplus Z \xrightarrow{(v,-u)} W$$

Since kernels exist in abelian categories, pullback must also exist.
Dually one shows pushouts exist. ∎

Lemma 23 $\mathcal{A}$ abelian cat, commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g\downarrow & & \downarrow v \\ Z & \xrightarrow{u} & W \end{array} \quad (*)$$

(1) If $(*)$ is a pullback, then the kernel morphism $\operatorname{Ker} f \to \operatorname{Ker} u$ is an iso

(2) If $(*)$ is a pushout, then the cokernel morphism $\operatorname{Coker} f \to \operatorname{Coker} u$ is an iso.

Pf: Prove (1), (2) is dual.

$$\begin{tikzcd} \mathrm{Ker} f \arrow[r, tail, "i_f"] \arrow[d, dashed, "h"] & X \arrow[r, "f"] \arrow[d, "g"] \arrow[dr, phantom, "(*)"] & Y \arrow[d, "v"] \\ \mathrm{Ker} u \arrow[r, tail, "i_u"] & Z \arrow[r, "u"'] & W \end{tikzcd}$$

h kernel morphism exists since
$u \circ g \circ i_f = v \circ f \circ i_f = 0$

h unique s.t. left hand square commutes.

Assume (*) pullback. Consider comm square

$$\begin{tikzcd} \mathrm{Ker}\, u \arrow[r, "0"] \arrow[d, "i_u"'] & Y \arrow[d, "v"] \\ Z \arrow[r, "u"] & W \end{tikzcd}$$

By universal property of pullback,
$\exists !$ morphism $k: \mathrm{Ker}\, u \longrightarrow X$ s.t.

$f \circ k = 0$, $g \circ k = i_u$

$$\begin{tikzcd}
\mathrm{Ker} f \arrow[r, tail, "i_f"] \arrow[d, "h"] & X \arrow[r, "f"] \arrow[d, "g"] \arrow[dr, phantom, "(*)"] & Y \arrow[d, "v"] \\
\mathrm{Ker} u \arrow[r, tail, "i_u"] \arrow[u, dashed, bend left, "\hat{k}"] \arrow[ur, dashed, "k"] & Z \arrow[r, "u"] & W
\end{tikzcd}$$

since $f \circ k = 0$, k factors through the kernel of f.

$\exists!\ \hat{k} : \mathrm{Ker}\,u \to \mathrm{Ker}\,f$ s.t. $i_f \circ \hat{k} = k$.

Want to show $\underset{(i)}{h \circ \hat{k} = id_{\mathrm{Ker}\,u}}$ & $\underset{(ii)}{\hat{k} \circ h = id_{\mathrm{Ker}\,f}}$

(i) Consider $i_u \circ h \circ \hat{k}$:

$$i_u \circ h \circ \hat{k} = g \circ i_f \circ \hat{k} = g \circ k = i_u = i_u \circ id_{\mathrm{Ker}\,u}$$

i_u monomorphism $\Rightarrow h \circ \hat{k} = id_{\mathrm{Ker}\,u}$

(ii) Since $(*)$ pullback, $\begin{pmatrix} f \\ g \end{pmatrix} : X \to Y \oplus Z$ mono by Lemma 21

$\Rightarrow \begin{pmatrix} f \\ g \end{pmatrix} \circ i_f$ mono.

Consider $\begin{pmatrix} f \\ g \end{pmatrix} \circ i_f \circ \hat{k} \circ h = \begin{pmatrix} f \\ g \end{pmatrix} \circ k \circ h$.

$$\begin{pmatrix} f \circ k \\ g \circ k \end{pmatrix} \circ h = \begin{pmatrix} 0 \\ i_u \end{pmatrix} \circ h = \begin{pmatrix} 0 \\ g \circ i_f \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \circ i_f$$

$\begin{pmatrix} f \\ g \end{pmatrix} \circ$ if mono $\Rightarrow \hat{k} \circ h = id_{\ker f}$

<u>Corollary</u> $\mathcal{A}$ abelian cat.

Consider commutative square
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "v"] \\ Z \arrow[r, "u"] & W \end{tikzcd} \quad (*)$$

The following hold:

(i) $(*)$ pushout & f epi $\Rightarrow$ u epi
(ii) $(*)$ pushout & f mono $\Rightarrow$ u mono
(iii) $(*)$ pullback & u mono $\Rightarrow$ f mono
(iv) $(*)$ pullback & u epi $\Rightarrow$ f epi

For (i) & (ii), say pushouts preserves monomorphisms/epimorphisms

(iii) & (iv), say pullbacks preserves monomorphisms/epimorphisms.

<u>Pf</u> (iii) & (iv) dual of (i) & (ii)

(i) follows immediately from Lemma ??,

since f epi $\Rightarrow \operatorname{Coker} f = 0 \Leftrightarrow \operatorname{Coker} u = 0 \Rightarrow u$ epi

(ii) If $(*)$ pushout and f mono, then $(*)$ pullback, by Corollary 2.2. Hence $\operatorname{Ker} f \cong \operatorname{Ker} u$.

f mono $\Rightarrow \operatorname{Ker} f \cong 0 \Leftrightarrow \operatorname{Ker} u \cong 0 \Rightarrow u$ mono ∎

Lecture 9

Some diagram lemmas

In ModR we can determine exactness using elements.

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

with $g \circ f = 0$ is exact if $\forall\, y \in Y$ such that $g(y) = 0$, there exists $x \in X$ s.t. $f(x) = y$.

Proofs are often easier when using elements. Things are more complicated for general abelian categories.

In this section we will only give proofs for ModR. However, all statements hold for general abelian categories.