Have comm diagrams

$$\begin{tikzcd}
 & X_1 \arrow[dl, "s_1"'] \arrow[dr, "\alpha"] & \\
X & & Y \\
 & X' \arrow[ul, "s"] \arrow[uu, "s_2'"] \arrow[ur, "\alpha \circ s_2'"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X_2 \arrow[dl, "s_2"'] \arrow[dr, "\beta"] & \\
X & & Y \\
 & X' \arrow[ul, "s"] \arrow[uu, "s_1'"] \arrow[ur, "\beta \circ s_1'"'] &
\end{tikzcd}$$

$\Rightarrow [s_1, \alpha] = [s, \alpha \circ s_2']$ & $[s_2, \beta] = [s, \beta \circ s_1']$

"common denominator"

$$[s_1, \alpha] + [s_2, \beta] = [s, \alpha \circ s_2'] + [s, \beta \circ s_1']$$
$$= [s, \alpha \circ s_2' + \beta \circ s_1'].$$

<u>Check</u>: This is well-defined

<u>Check</u>: This makes $\mathrm{Hom}_{S^{-1}\mathcal{B}}(X, Y)$ into an abelian group, with neutral element

$(X \xleftarrow{1} X \xrightarrow{0} Y)$

<u>Check</u>: composition in $S^{-1}\mathcal{B}$ is bilinear

$\Rightarrow S^{-1}\mathcal{B}$ is preadditive.

Clearly, with this definition, the functor $\mathcal{B} \xrightarrow{q} S^{-1}\mathcal{B}$ is additive.

remains to show: $S^{-1}\mathcal{B}$ - has a zero obj

- has biproducts.

Claim: $q(0)$ zero obj in $S^{-1}\mathcal{C}$, i.e. initial and terminal:

initial: Let $0 \xleftarrow{s} X \xrightarrow{\alpha} Y$ arbitrary

Can find comm diagram

$$\begin{tikzcd}
& X \arrow[dl, "s"'] \arrow[dr, "\alpha"] & \\
0 & & Y \\
& 0 \arrow[ul, "1_0"] \arrow[uu, "0"] \arrow[ur, "0"'] &
\end{tikzcd}$$

$\Rightarrow [s,\alpha] = [1,0]$

$\Rightarrow$ unique morphism

$\Rightarrow 0$ initial in $S^{-1}\mathcal{B}$.

0 being terminal is proved similarly

Finally, since $\mathcal{B}$ has biproducts and $q: \mathcal{B} \to S^{-1}\mathcal{B}$ additive and bijective on objects $\Rightarrow S^{-1}\mathcal{B}$ has biproducts (why?)

$\Rightarrow S^{-1}\mathcal{B}$ additive ■

Corollary

$D(\mathcal{A})$ is additive ■

Lemma: Let $X^\bullet \in K(\mathcal{A})$. Then

$X^\bullet \cong 0$ in $D(\mathcal{A}) \iff X^\bullet$ is exact

Proof: exercise ∎

Lemma: A morphism $f: X^\bullet \to Y^\bullet$ in $K(\mathcal{A})$ gets sent to 0 in $D(\mathcal{A})$ iff $f^\bullet$ factors through an exact complex.

Proof: Exercise ∎

Given a quasi-iso q, the shift $q[n]$ is also a quasi-iso. Hence the composite

$$K(\mathcal{A}) \xrightarrow{[n]} K(\mathcal{A}) \to D(\mathcal{A})$$

sends quasi-isomorphisms to isomorphisms. Hence there exists a unique functor $[n]: D(\mathcal{A}) \to D(\mathcal{A})$ s.t. the diagram

$$\begin{tikzcd} K(\mathcal{A}) \arrow[r, "{[n]}"] \arrow[d] & K(\mathcal{A}) \arrow[d] \\ D(\mathcal{A}) \arrow[r, "{[n]}"] & D(\mathcal{A}) \end{tikzcd}$$

commutes.

For a morphism $f^\bullet: X^\bullet \to Y^\bullet$ of complexes we also write $f^\bullet$ for the corresponding morphism $(X^\bullet, [f^\bullet], Y^\bullet)$ in $D(\mathcal{A})$

Theorem: $(D(\mathcal{A}), [1], \Delta)$ is a triangulated category where Δ consists of all triangles in $D(\mathcal{A})$ isomorphic in $D(\mathcal{A})$ to standard triangles

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \to \mathrm{Cone}(f^\bullet) \to X^\bullet[1]$$

where $f^\bullet$ is a morphism of complexes.

Proof: See Steffen's notes ∎

We have more triangles in $D(\mathcal{A})$ than in $K(\mathcal{A})$!

Proposition: Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence in $\mathcal{A}$. Then there is a distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \to A[1]$ in $D(\mathcal{A})$.

(Here we use the same notation for objects & morphisms in $\mathcal{A}$ & their image under the functor $\mathcal{A} \to Ch(\mathcal{A}) \to K(\mathcal{A}) \to D(\mathcal{A})$.)

Proof: Consider the standard triangle

$$A \xrightarrow{f} B \xrightarrow{\binom{1}{0}} \mathrm{Cone}(f) \xrightarrow{(0,1)} A[1] \quad (*)$$

Here

$$\begin{tikzcd}
 & \text{degree} & -2 & -1 & 0 & 1 & \\
\mathrm{Cone}(f) \arrow[d, "q"] & = \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r, "f"] \arrow[d] & B \arrow[r] \arrow[d, "g"] & 0 \arrow[r] \arrow[d] & \cdots \\
C & = \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & C \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

q is a quasi-isomorphism (why?)
Hence $(*)$ is isomorphic in $D(\mathcal{A})$ to the triangle

$$A \xrightarrow{f} B \xrightarrow{q \circ \binom{1}{0}} C \xrightarrow{(0,1) \circ q^{-1}} A[1]$$

Since $q \circ \binom{1}{0} = g$, the claim follows ∎

Last goal: realize $\mathrm{Ext}^n_{\mathcal{A}}(A, B)$ as a Hom space in $D(\mathcal{A})$.

<u>Theorem</u>: Let $\mathcal{A}$ be an abelian cat, and $P^{\cdot}$ a right bounded complex of projectives (i.e. P^n proj $\forall n \in \mathbb{Z}$, & $\exists N$ s.t. $P^n = 0$ for $n > N$

(1) If $E^\bullet$ is an exact complex, then
$\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, E^\bullet) = 0$

(2) If $s: X^\bullet \to Y^\bullet$ is a quasi-isomorphism, then $\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, s): \mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, X^\bullet) \to \mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, Y^\bullet)$
is an isomorphism

(3) For any complex $X^\bullet$ the map
$\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, X^\bullet) \to \mathrm{Hom}_{D(\mathcal{A})}(P^\bullet, X^\bullet)$
is an isomorphism.

<u>Proof</u>:
(1) $f^\bullet: P^\bullet \to E^\bullet$ morphism in $Ch(\mathcal{A})$.
Construct null-homotopy $h^\bullet$ of $f^\bullet$ iteratively from right to left
Assume have $h^i: P^i \to E^{i-1}$ for $i > n$ s.t.
$f^i = d_E^{i-1} \circ h^i + h^{i+1} \circ d_P^i$. (note: $h^i = 0$ for $i > N$)

$$
\begin{tikzcd}
 & & P^n \arrow[r, "d_P^n"] \arrow[dll, dashed, "h^n"'] \arrow[d, "f^n"] \arrow[ddl, dashed] & P^{n+1} \arrow[dl, "h^{n+1}"'] \arrow[d, "f^{n+1}"] \\
E^{n-1} \arrow[rr, "d_E^{n-1}"] \arrow[dr, two heads] & & E^n \arrow[r, "d_E^n"'] & E^{n+1} \\
 & \mathrm{Im}\, d_E^{n-1} \arrow[ur, hook] & &
\end{tikzcd}
$$

$d_E^n \circ (f^n - h^{n+1} \circ d_P^n) = 0$

$\Rightarrow f^n - h^{n+1} \circ d_P^n$ factors through $\mathrm{Im}\, d_E^{n-1} \rightarrowtail E^n$

P^n projective $\Rightarrow P^n \to \mathrm{Im}\, d_E^{n-1}$ lift along epi $E^{n-1} \twoheadrightarrow \mathrm{Im}\, d_E^{n-1}$ to a morphism $h^n : P^n \longrightarrow E^{n-1}$. Then $d_E^{n-1} \circ h^n = f^n - h^{n+1} \circ d_P^n$

$\Leftrightarrow f^n = d_E^{n-1} \circ h^n + h^{n+1} \circ d_P^n$

(2) $s^\bullet$ quasi-iso $\Leftrightarrow$ $\mathrm{cone}(s^\bullet)$ exact.
Apply $\mathrm{Hom}_{K(\mathcal{C})}(P^\bullet, -)$ to triangle

$$X^\bullet \xrightarrow{s^\bullet} Y^\bullet \to \mathrm{Cone}(s^\bullet) \to X^\bullet[1]$$

get exact seq