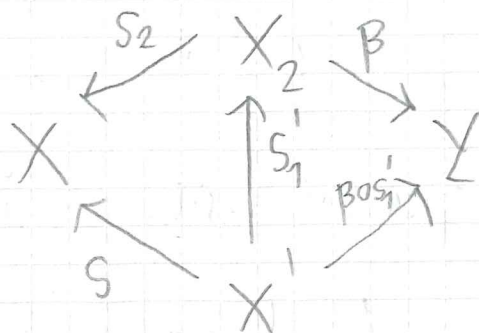
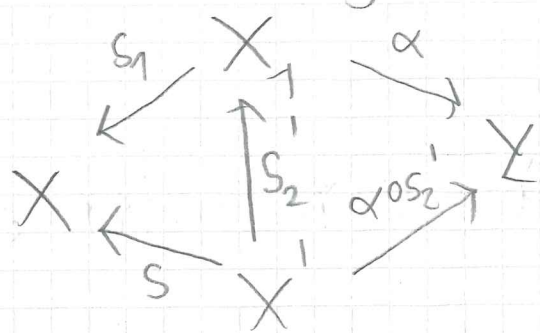


Have comm diagrams



$$\Rightarrow [s_1, \alpha] = [s, \alpha \circ s_2'] \quad \& \quad [s_2, \beta] = [s, \beta \circ s_1']$$

"common denominator"

$$[s_1, \alpha] + [s_2, \beta] = [s, \alpha \circ s_2'] + [s, \beta \circ s_1'] \\ = [s, \alpha \circ s_2' + \beta \circ s_1']$$

Check: This is well-defined

Check: This makes $\text{Hom}_{\mathcal{S}\mathcal{B}}^{-1}(X, Y)$ into an abelian group, with neutral element

$$\left(X \xleftarrow{1} X \xrightarrow{0} Y \right)$$

Check: composition in $\mathcal{S}\mathcal{B}^{-1}$ is bilinear
 $\Rightarrow \mathcal{S}\mathcal{B}^{-1}$ is preadditive.

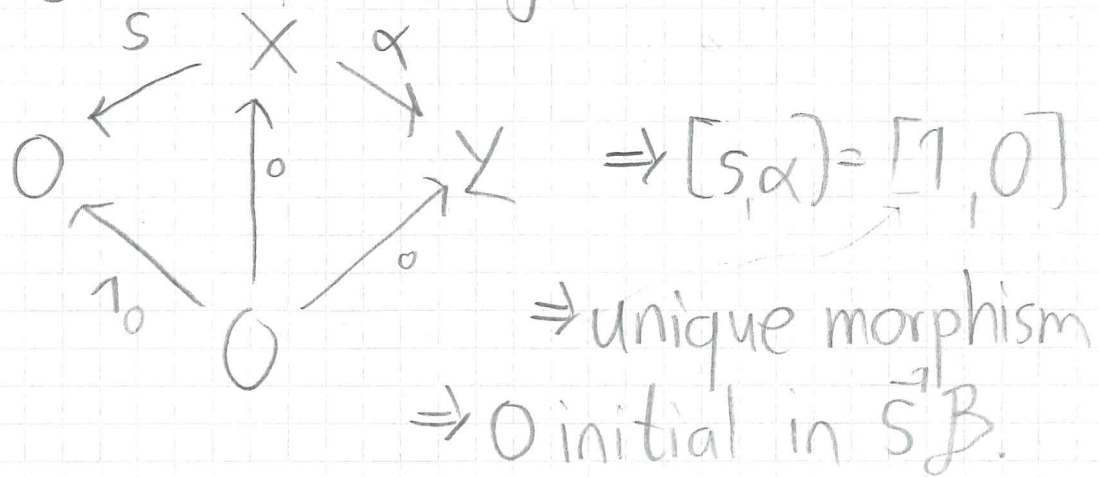
Clearly, with this definition, the functor $\mathcal{B} \xrightarrow{q} \mathcal{S}\mathcal{B}^{-1}$ is additive.

remains to show: $\mathcal{S}\mathcal{B}^{-1}$ has a zero obj
- has biproducts.

Claim: $q(0)$ zero obj in $\tilde{S}^1 \mathcal{C}$, i.e. initial and terminal:

initial: let $0 \xleftarrow{s} X \xrightarrow{\alpha} Z$ arbitrary

Can find comm diagram



0 being terminal is proved similarly

Finally, since \mathcal{B} has biproducts and $q: \mathcal{B} \rightarrow \tilde{S}^1 \mathcal{B}$ additive and bijective on objects $\Rightarrow \tilde{S}^1 \mathcal{B}$ has biproducts (why?)
 $\Rightarrow \tilde{S}^1 \mathcal{B}$ additive \blacksquare

Corollary

$D(\mathcal{A})$ is additive \blacksquare

Lemma: Let $X' \in K(\mathcal{A})$. Then

$X' \cong 0$ in $D(\mathcal{A}) \Leftrightarrow X'$ is exact

Proof: exercise ▀

Lemma: A morphism $f: X^\bullet \rightarrow Y^\bullet$ in $K(\mathcal{A})$ gets sent to 0 in $D(\mathcal{A})$ iff f factors through an exact complex.

Proof: Exercise ▀

Given a quasi-iso q , the shift $q[n]$ is also a quasi-iso. Hence the composite

$$K(\mathcal{A}) \xrightarrow{[n]} K(\mathcal{A}) \rightarrow D(\mathcal{A})$$

sends quasi-isomorphisms to isomorphisms, hence there exists a unique functor

$[n]: D(\mathcal{A}) \rightarrow D(\mathcal{A})$ s.t. the diagram

$$\begin{array}{ccc} K(\mathcal{A}) & \xrightarrow{[n]} & K(\mathcal{A}) \\ \downarrow & & \downarrow \\ D(\mathcal{A}) & \xrightarrow{[n]} & D(\mathcal{A}) \end{array}$$

commutes

For a morphism $f: X^\bullet \rightarrow Y^\bullet$ of complexes we write f for the corresponding morphism $(X^\bullet, [f], Y^\bullet)$ in $D(\mathcal{A})$

Theorem: $(D(\mathcal{A}), E[1], \Delta)$ is a triangulated category where Δ consists of all triangles in $D(\mathcal{A})$ isomorphic in $D(\mathcal{A})$ to standard triangles

$$X \xrightarrow{f} Y \rightarrow \text{Cone}(f) \rightarrow X[1]$$

where f is a morphism of complexes.

Proof: See Steffen's notes.

We have more triangles in $D(\mathcal{A})$ than in $K(\mathcal{A})$!

Proposition: Let $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be an exact sequence in \mathcal{A} . Then there is a distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \rightarrow A[1]$ in $D(\mathcal{A})$.

(Here we use the same notation for objects & morphisms in \mathcal{A} & their image under the functor $\mathcal{A} \rightarrow \text{Ch}(\mathcal{A}) \rightarrow K(\mathcal{A}) \rightarrow D(\mathcal{A})$.)

Proof: Consider the standard triangle

$$A \xrightarrow{f} B \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \text{Cone}(f) \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} A[1] \quad (*)$$

Here

$$\begin{array}{ccccccc} & \text{degree} & -2 & -1 & 0 & 1 & \\ \text{Cone}(f) = \dots & \rightarrow & 0 & \rightarrow & A & \xrightarrow{f} & B & \rightarrow & 0 & \rightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ & & q & & & & g & & & \\ C = \dots & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & C & \rightarrow & 0 & \rightarrow \dots \end{array}$$

q is a quasi-isomorphism (why?).
Hence $(*)$ is isomorphic in $D(\mathcal{A})$ to the triangle

$$A \xrightarrow{f} B \xrightarrow{q \circ \begin{pmatrix} 1 \\ 0 \end{pmatrix}} C \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix} \circ q^{-1}} A[1]$$

Since $q \circ \begin{pmatrix} 1 \\ 0 \end{pmatrix} = g$, the claim follows. \blacksquare

Last goal: realize $\text{Ext}_{\mathcal{A}}^n(A, B)$ as a Hom-space in $D(\mathcal{A})$.

Theorem: Let \mathcal{A} be an abelian cat,
and P^\bullet a right bounded complex of projectives
(i.e. P^n proj $\forall n \in \mathbb{Z}$, & $\exists N$ s.t. $P^n = 0$ for $n > N$)

(1) If E^\bullet is an exact complex, then

$$\text{Hom}_{\text{K}(\mathcal{A})}(P^\bullet, E^\bullet) = 0$$

(2) If $s: X^\bullet \rightarrow Y^\bullet$ is a quasi-isomorphism, then $\text{Hom}_{\text{K}(\mathcal{A})}(P^\bullet, s): \text{Hom}_{\text{K}(\mathcal{A})}(P^\bullet, X^\bullet) \rightarrow \text{Hom}_{\text{K}(\mathcal{A})}(P^\bullet, Y^\bullet)$ is an isomorphism

(3) For any complex X^\bullet the map $\text{Hom}_{\text{K}(\mathcal{A})}(P^\bullet, X^\bullet) \rightarrow \text{Hom}_{\text{D}(\mathcal{A})}(P^\bullet, X^\bullet)$ is an isomorphism.

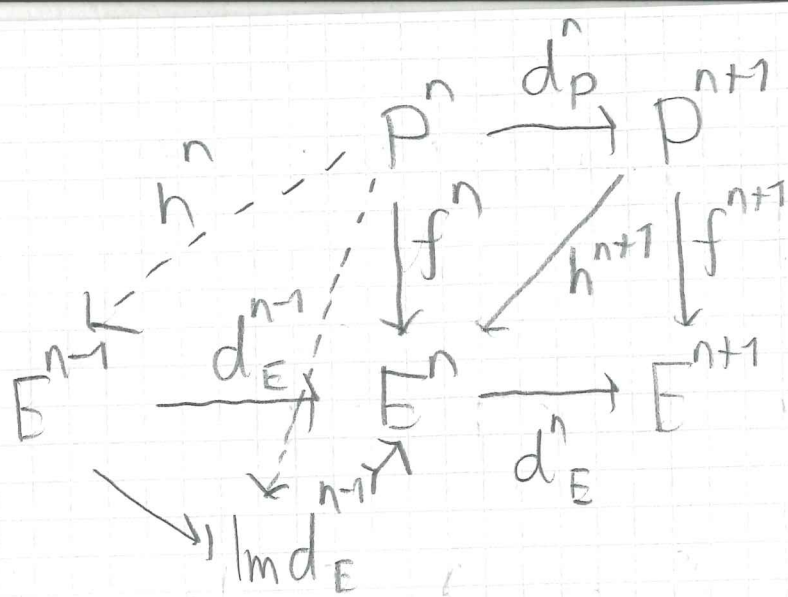
Proof:

(1) $f: P^\bullet \rightarrow E^\bullet$ morphism in $\text{Ch}(\mathcal{A})$.

Construct null-homotopy h^\bullet of f^\bullet iteratively from right to left

Assume have $h^i: P^i \rightarrow E^{i-1}$ for $i > n$ s.t.

$$f^i = d_E^{i-1} \circ h^i + h^{i+1} \circ d_P^i \quad (\text{note: } h^i = 0 \text{ for } i > N)$$



$$d_E^n \circ (f^n - h^{n+1} \circ d_P^n) = 0$$

$\Rightarrow f^n - h^{n+1} \circ d_P^n$ factors through $\text{Im } d_E^{n-1} \rightarrow E^n$

P^n projective $\Rightarrow P^n \rightarrow \text{Im } d_E^{n-1}$ lift along
 epi $E^{n-1} \rightarrow \text{Im } d_E^{n-1}$ to a morphism
 $h^n: P^n \rightarrow E^{n-1}$. Then $d_E^n \circ h^n = f^n - h^{n+1} \circ d_P^n$

$$\Leftrightarrow f^n = d_E^n \circ h^n + h^{n+1} \circ d_P^n$$

(2) s quasi-iso $\Leftrightarrow \text{con}(s)$ exact.

Apply $\text{Hom}_{K(\mathbb{C})}(P_i, -)$ to triangle

$$X' \xrightarrow{s} Y' \rightarrow \text{con}(s) \rightarrow X'[1]$$

get exact seq