

But we know $h = i \circ g = i \circ h$, so
 by uniqueness $g = h$.

Lecture 7 Abelian categories

Proposition: \mathcal{A} additive category,
 $f: X \rightarrow Y$ morphism in \mathcal{A} . Assume
 $\ker f$ & $\operatorname{Coker} f$ exists

$$\ker f \xrightarrow{i} X \xrightarrow{f} Y \xrightarrow{p} \operatorname{Coker} f$$

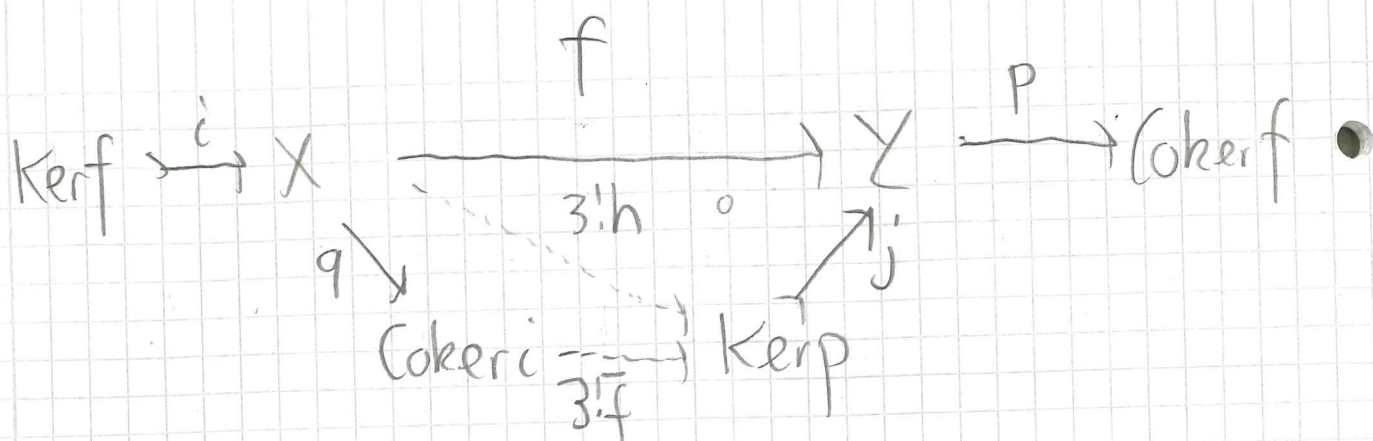
Assume also that $\operatorname{Coker} i$ & $\ker p$ exists

Then \exists unique map $\bar{f}: \operatorname{Coker} i \rightarrow \ker p$
 s.t. the diagram

$$\begin{array}{ccccc} \ker f & \xrightarrow{i} & X & \xrightarrow{f} & Y & \xrightarrow{p} & \operatorname{Coker} f \\ & & \searrow q & & \nearrow j & & \\ & & \operatorname{Coker} i & \xrightarrow{\bar{f}} & \ker p & & \end{array}$$

commutes.

Pf: Since $p \circ f = 0$, \exists unique map
 $h: X \rightarrow \ker p$ s.t. $j \circ h = f$.



$(\text{Ker } p, j)$ kernel of $p \Rightarrow j$ monomorphism.

$$j \circ h \circ i = f \circ i = 0 = j \circ 0$$

$$j \text{ mono} \Rightarrow h \circ i = 0$$

$\Rightarrow \exists$ unique $\bar{f}: \text{Coker } i \rightarrow \text{Ker } p$

s.t. $q \circ \bar{f} = h$. Hence

$$j \circ \bar{f} \circ q = j \circ h = f.$$

Since h unique s.t. $j \circ h = f$

\bar{f} unique s.t. $q \circ \bar{f} = h$

$\Rightarrow \bar{f}$ unique s.t. $j \circ \bar{f} \circ q = f$.

Def: In the diagram above,

• $\text{Coker } i$ called the coimage of f
denoted $\text{Coim } f$

• $\text{Ker } p$ called the image of f ,
denoted $\text{Im } f$

Def: \mathcal{A} additive cat.

- \mathcal{A} is preabelian if every morphism in \mathcal{A} has a kernel & a cokernel
- \mathcal{A} is abelian if it is preabelian and the map $\bar{f}: \text{Coim}f \rightarrow \text{Im}f$ is an isomorphism for every morphism f in \mathcal{A} .

Example

If $\mathcal{A} = \text{Mod } R$, then every morphism $f: M \rightarrow N$ has a kernel ($\text{Ker}f = \{m \in M \mid f(m) = 0\}$) and a cokernel ($\text{Coker}f = \frac{N}{\{f(m) \mid m \in M\}}$)

$\Rightarrow \text{Mod } R$ preabelian.

Image of f : $\text{Im}f = \{f(m) \mid m \in M\}$

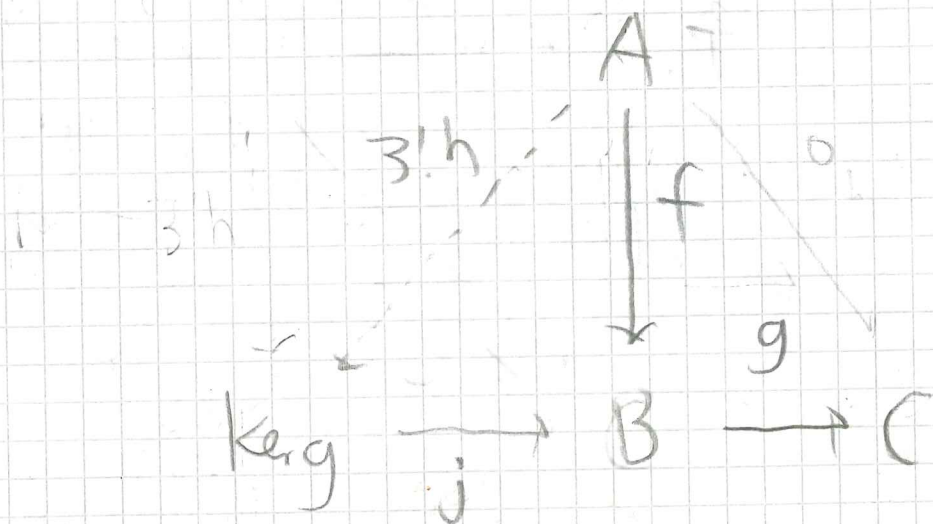
Coimage of f : $\text{Coim}f = \frac{M}{\text{Ker}f}$

So the isomorphism $\text{Coim}f \rightarrow \text{Im}f$ just amounts to the first isomorphism theorem!

Exact sequences, pullbacks and pushouts

\mathcal{A} abelian: Will identify image and coimage in \mathcal{A}

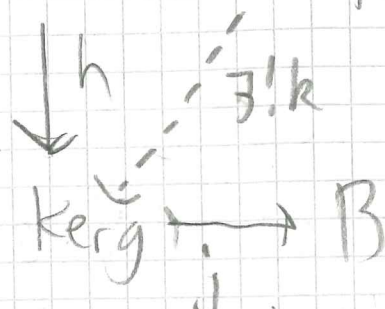
Let $A \xrightarrow{f} B \xrightarrow{g} C$ s.t. $g \circ f = 0$



$g \circ f = 0 \Rightarrow \exists$ unique $h: A \rightarrow \text{Ker } g$
s.t. $j \circ h = f$.

$\text{Ker } f \xrightarrow{i} A \xrightarrow{p} \text{Im } f = \text{Coker } i$

j mono
& $j \circ h \circ i = f \circ i = 0$
 $\Rightarrow h \circ i = 0$



$\Rightarrow \exists$ unique map k s.t. $h = k \circ p$

have comm diagram

$$\begin{array}{ccccc} \text{Im } f & \longrightarrow & B & \longrightarrow & \text{Coker } f \\ \downarrow k & & \parallel & \circ & \downarrow \exists! \ell \\ \text{Ker } g & \longrightarrow & B & \longrightarrow & \text{Im } g \end{array}$$

ℓ exists by universal property of $\text{Coker } f$.

Can show: k isomorphism $\Leftrightarrow \ell$ isomorphism.

Def: A sequence $A \xrightarrow{f} B \xrightarrow{g} C$
with $g \circ f = 0$ is exact if
 $k: \text{Im } f \rightarrow \text{Ker } g$ is an iso

A sequence $(\cdots \xrightarrow{f_{n-2}} A_{n-1} \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \xrightarrow{f_{n+1}} \cdots)$
is exact if $A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1}$
is exact $\forall i \in \mathbb{Z}$.

Lecture 8

Rmk: By abuse of notation sometimes

say $\text{Im } f = \text{Ker } g$ (even though they are only iso)

(2) (i) $0 \rightarrow A \xrightarrow{f} B$ exact $\Leftrightarrow \text{Ker } f = 0$
 $\Leftrightarrow f$ mono