But we know $k = i \circ g = i \circ h$, so by uniqueness $\ell = g = h$.

Lecture 7

Abelian categories

Proposition: $\mathcal{A}$ additive category, $f: X \to Y$ morphism in $\mathcal{A}$. Assume $\ker f$ & $\operatorname{Coker} f$ exists

$$\ker f \xrightarrow{i} X \xrightarrow{f} Y \xrightarrow{p} \operatorname{Coker} f$$

Assume also that $\operatorname{Coker} i$ & $\ker p$ exists. Then $\exists$ unique map $\bar{f}: \operatorname{Coker} i \to \ker p$ s.t. the diagram

$$\begin{tikzcd}
\ker f \arrow[r, tail, "i"] & X \arrow[rr, "f"] \arrow[d, "g"'] & & Y \arrow[r, two heads, "p"] & \operatorname{Coker} f \\
& \operatorname{Coker} i \arrow[r, dashed, "\bar{f}"] & \ker p \arrow[ur, "j"'] & &
\end{tikzcd}$$

commutes.

Pf: Since $p \circ f = 0$, $\exists$ unique map $h: X \to \ker p$ s.t. $j \circ g = f$.

$$
\begin{tikzcd}
\operatorname{Ker} f \arrow[r, "i"] & X \arrow[rr, "f"] \arrow[d, "q"'] \arrow[drr, dashed, "\exists! h"] & & Y \arrow[r, "p"] & \operatorname{Coker} f \\
& \operatorname{Coker} i \arrow[rr, dashed, "\exists! \bar{f}"'] & & \operatorname{Ker} p \arrow[u, "j"'] &
\end{tikzcd}
$$

$(\operatorname{Ker} p, j)$ kernel of $p \Rightarrow j$ monomorphism.

$j \circ h \circ i = f \circ i = 0 = j \circ 0$

j mono $\Rightarrow h \circ i = 0$

$\Rightarrow \exists$ unique $\bar{f} : \operatorname{Coker} i \to \operatorname{Ker} p$

s.t. $q \circ \bar{f} = h$. Hence

$j \circ \bar{f} \circ q = j \circ h = f$.

Since h unique s.t. $j \circ h = f$

$\bar{f}$ unique s.t. $q \circ \bar{f} = h$

$\Rightarrow \bar{f}$ unique s.t. $j \circ \bar{f} \circ q = f$ ∎

Def: In the diagram above,

- $\operatorname{Coker} i$ called the coimage of f denoted $\operatorname{Coim} f$
- $\operatorname{Ker} p$ called the image of f. denoted $\operatorname{Im} f$

Def: $\mathcal{A}$ additive cat.

- $\mathcal{A}$ is prealbelian if every morphism in $\mathcal{A}$ has a kernel & a cokernel
- $\mathcal{A}$ is abelian if it is prealbelian and the map $\overline{f}: \mathrm{Coim} f \to \mathrm{Im} f$ is an isomorphism for every morphism f in $\mathcal{A}$.

Example

If $\mathcal{A} = \mathrm{Mod}\, R$, then every morphism $f: M \to N$ has a kernel ($\mathrm{Ker} f = \{m \in M \mid f(m) = 0\}$) and a cokernel ($\mathrm{Coker} f = N / \{f(m) \mid m \in M\}$)

$\Rightarrow \mathrm{Mod}\, R$ prealbelian.

Image of f: $\mathrm{Im} f = \{f(m) \mid m \in M\}$

Coimage of f: $\mathrm{Coim} f = M / \mathrm{Ker} f$

So the isomorphism $\mathrm{CoIm} f \to \mathrm{Im} f$ just amounts to the first isomorphism theorem!

Exact sequences, pullbacks and pushouts

$\mathcal{A}$ abelian: Will identify image and coimage in $\mathcal{A}$

Let $A \xrightarrow{f} B \xrightarrow{g} C$ s.t. $g \circ f = 0$

$$\begin{tikzcd}
 & A \arrow[dl, dashed, "\exists! h"'] \arrow[d, "f"] \arrow[dr, "0"] & \\
\ker g \arrow[r, "j"'] & B \arrow[r, "g"] & C
\end{tikzcd}$$

$g \circ f = 0 \Rightarrow \exists$ unique $h : A \to \ker g$ s.t. $j \circ h = f$.

$$\begin{tikzcd}
\operatorname{Ker} f \arrow[r, tail, "i"] & A \arrow[r, "p"] \arrow[d, "h"] & \operatorname{Im} f = \operatorname{Coker} i \arrow[dl, dashed, "\exists! k"] \\
 & \ker g \arrow[r, tail, "j"'] & B
\end{tikzcd}$$

j mono

& $j \circ h \circ i = f \circ i = 0$

$\Rightarrow h \circ i = 0$

$\Rightarrow \exists$ unique map k s.t. $h = k \circ p$

have comm diagram

$$\begin{tikzcd} \mathrm{Im} f \arrow[r, tail] \arrow[d, "k"] & B \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circ"] & \mathrm{Coker} f \arrow[d, dashed, "\exists ! \ell"] \\ \ker g \arrow[r, tail] & B \arrow[r, two heads] & \mathrm{Im} g \end{tikzcd}$$

ℓ exists by universal property of Coker f.

Can show: k isomorphism $\Leftrightarrow$ ℓ isomorphism.

<u>Def</u>: • A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ with $g \circ f = 0$ is <u>exact</u> if $k : \mathrm{Im} f \to \ker g$ is an iso

• A sequence $(\cdots \xrightarrow{f_{n-2}} A_{n-1} \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \xrightarrow{f_{n+1}} \cdots)$ is <u>exact</u> if $A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1}$ is exact $\forall i \in \mathbb{Z}$

<u>Lecture 8</u>

<u>Rmk</u>: By abuse of notation sometimes say $\mathrm{Im} f = \ker g$ (even though they are only iso)

(2) (i) $0 \to A \xrightarrow{f} B$ exact $\Leftrightarrow \ker f \cong 0$
$\Leftrightarrow f$ mono