MA3204 - Exercise 7

Throughout the following exercises $\mathcal{A}$ denotes an abelian category, and $\mathcal{C}$ denotes an arbitrary category. For the exercises about localizations we will ignore set-theoretical issues.

1. Let $\mathcal{A}$ be an additive category, let $X \in \mathcal{A}$, and let $A^\bullet = (\cdots \xrightarrow{d^{-2}} A^{-1} \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots)$ be a complex in $\mathcal{A}$. Define the complex

   $$\operatorname{Hom}_{\mathcal{A}}(A^\bullet, X) = (\cdots \xrightarrow{-\circ d^1} \operatorname{Hom}_{\mathcal{A}}(A^1, X) \xrightarrow{-\circ d^0} \operatorname{Hom}_{\mathcal{A}}(A^0, X) \xrightarrow{-\circ d^{-1}} \operatorname{Hom}_{\mathcal{A}}(A^{-1}, X) \xrightarrow{-\circ d^{-2}} \cdots)$$

   Show that

   $$Z^n \operatorname{Hom}_{\mathcal{A}}(A^\bullet, X) = \operatorname{Hom}_{\mathbf{Ch}(\mathcal{A})}(A^\bullet, X[n])$$
   $$H^n \operatorname{Hom}_{\mathcal{A}}(A^\bullet, X) = \operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(A^\bullet, X[n])$$

   Hint: Show that a morphism $f^{-n}\colon A^{-n} \to X$ induces a morphism $A^\bullet \to X[n]$ if and only if f^{-n} is in the kernel of

   $$\operatorname{Hom}_{\mathcal{A}}(A^{-n}, X) \xrightarrow{-\circ d^{-n-1}} \operatorname{Hom}_{\mathcal{A}}(A^{-n-1}, X)$$

   and is null-homotopic if and only if it is in the image of

   $$\operatorname{Hom}_{\mathcal{A}}(A^{-n+1}, X) \xrightarrow{-\circ d^{-n}} \operatorname{Hom}_{\mathcal{A}}(A^{-n}, X)$$

2. Let $\mathcal{C}$ be a category, and let $S \subset \operatorname{Mor}\mathcal{C}$ be a class of morphisms in $\mathcal{C}$. Consider the localization $\mathcal{C}[S^{-1}]$ defined in the lecture, whose morphism spaces consists of strings of morphisms in $\mathcal{C}$ and formal inverses of morphisms in S.

   (a) Show that $\mathcal{C}[S^{-1}]$ is a category, with identity morphisms given by the empty strings $[X, \emptyset, X]$.

   (b) Show that we have a functor $Q\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ given by the identity on objects, and sending a morphism $f\colon X \to Y$ to the string $Q(f) = [X, f, Y]$. Furthermore, show that if $s \in S$, then $Q(s)$ is an isomorphism

   (c) Show that $(\mathcal{C}[S^{-1}], Q)$ satisfies the universal property of the localization. In other words, if $F\colon \mathcal{C} \to \mathcal{D}$ is a functor satisfying that $F(s)$ is an isomorphism for all $s \in S$, then show that there exists a unique functor

   $$\overline{F}\colon \mathcal{C}[S^{-1}] \to \mathcal{D}$$

   satisfying $F = \overline{F} \circ Q$.

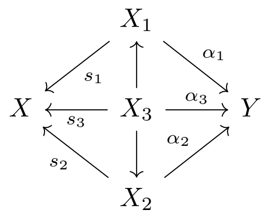
   Hint: For the last part, $\overline{F}$ is defined by sending an object C to $F(C)$, and by sending a string $[X, p_n, p_{n-1}, \ldots, p_1, Y]$ to the morphism

   $$\overline{F}(p_n) \circ \overline{F}(p_{n-1}) \circ \cdots \circ \overline{F}(p_1)$$

   where $\overline{F}(p_i) = F(p_i)$ if p_i is a morphism in $\mathcal{C}$, and $\overline{F}(p_i) = F(s)^{-1}$ if $p_i = s^- \in S^-$ for a morphism $s \in S$.

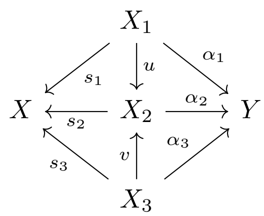
3. Let $\mathcal{C}$ be a category, and let $S \subset \operatorname{Mor}\mathcal{C}$ be a class of morphisms in $\mathcal{C}$. Assume S admits a calculus of right fractions. We defined two right fractions (s_1, α_1) and (s_2, α_2) from X to Y to be

equivalent, written $(s_1, \alpha_1) \sim (s_2, \alpha_2)$, if there exists a right fraction (s_3, α_3) and a commutative diagram

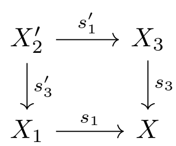


Show that $\sim$ is an equivalence relation on the class of right fractions from X to Y. We denote the equivalence class of (s, α) by $[s, \alpha]$.

Hint: For transitivity, show first that if (s_1, α_1), (s_2, α_2) and (s_3, α_3) are three right fractions related by the commutative diagram

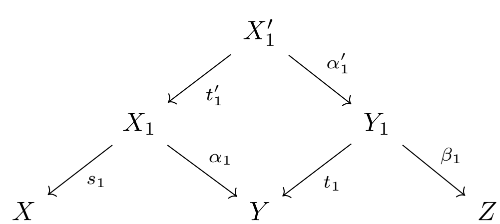


then $(s_1, \alpha_1) \sim (s_3, \alpha_3)$. For this, first apply the (RF2) axiom to $X_3 \xrightarrow{s_3} X \xleftarrow{s_1} X_1$ to get a commutative square



with $s_1' \in S$. This gives two morphisms $X_2' \xrightarrow{u \circ s_3'} X_2$ and $X_2' \xrightarrow{v \circ s_1'} X_2$ which becomes equal when composing with $s_2 \in S$. Now apply axiom (RF3), and then conclude that $(s_1, \alpha_1) \sim (s_3, \alpha_3)$.

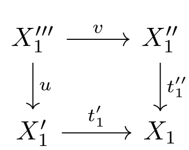
4. Let $\mathcal{C}$ be a category, and let $S \subset \operatorname{Mor} \mathcal{C}$ be a class of morphisms in $\mathcal{C}$. Assume S admits a calculus of right fractions. Recall that a composite of two right fractions $X \xleftarrow{s_1} X_1 \xrightarrow{\alpha_1} Y$ and $Y \xleftarrow{t_1} Y_1 \xrightarrow{\beta_1} Z$ is a right fraction $(s_1 \circ t_1', \beta_1 \circ \alpha_1')$ where $t_1' \in S$ and where we have commutative square



obtained from axiom (RF2).

(a) Show that the equivalence class of $(s_1 \circ t_1', \beta_1 \circ \alpha_1')$ is independent of the choice of morphisms t_1' and α_1' above. Let $(t_1, \beta_1) \circ (s_1, \alpha_1)$ denote this unique class.

Hint: Assume (t_1'', α_1'') is another pair of morphisms satisfying the same conditions as (t_1', α_1'). Using (RF2), we can find a commutative square

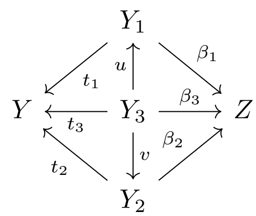


with $u \in S$. Then apply axiom (RF3) to conclude that $(s_1 \circ t_1', \beta_1 \circ \alpha_1') \sim (s_1 \circ t_1''', \beta_1 \circ \alpha_1'')$

(b) Show that if $(t_1, \beta_1) \sim (t_2, \beta_2)$, then

$$(t_1, \beta_1) \circ (s_1, \alpha_1) = (t_2, \beta_2) \circ (s_1, \alpha_1)$$

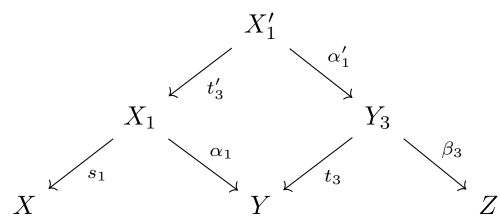
Hint: $(t_1, \beta_1) \sim (t_2, \beta_2)$ *means there exists a commutative diagram*



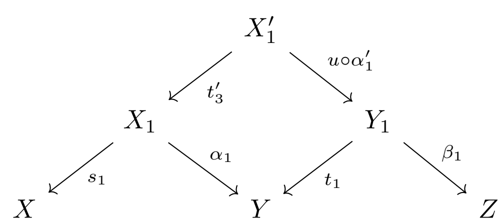
with $t_3 \in S$*. Hence, if we can show that*

$$(t_1, \beta_1) \circ (s_1, \alpha_1) = (t_3, \beta_3) \circ (s_1, \alpha_1)$$

then a similar argument will also imply that $(t_2, \beta_2) \circ (s_1, \alpha_1) = (t_3, \beta_3) \circ (s_1, \alpha_1)$*, and we can conclude using the transitivity of the equivalence relation.*
To see that $(t_2, \beta_2) \circ (s_1, \alpha_1) = (t_3, \beta_3) \circ (s_1, \alpha_1)$*, choose a commutative diagram*



with $t_3' \in S$*. Then* $(t_3, \beta_3) \circ (s_1, \alpha_1) = [s_1 \circ t_3', \beta_3 \circ \alpha_1']$*. Conclude by considering the commutative diagram*



and use that $\beta_1 \circ u \circ \alpha_1' = \beta_3 \circ \alpha_1'$*.*

(c) Show that if $(s_1, \alpha_1) \sim (s_2, \alpha_2)$, then

$$(t_1, \beta_1) \circ (s_1, \alpha_1) = (t_1, \beta_1) \circ (s_2, \alpha_2)$$

(d) Consider $S^{-1}\mathcal{C}$ as defined in the lecture, whose morphisms spaces are equivalence classes of right fractions. Show that the composition law above induces a well-defined map

$$\circ \colon \mathrm{Hom}_{S^{-1}\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{S^{-1}\mathcal{C}}(X, Y) \to \mathrm{Hom}_{S^{-1}\mathcal{C}}(X, Z).$$

Hint: Combine part (b) and (c) and the fact that the equivalence relation is transitive.

5. Let $\mathcal{C}$ be a category, and let $S \subset \mathrm{Mor}\,\mathcal{C}$ be a class of morphisms in $\mathcal{C}$. Assume S admits a calculus of right fractions. Using the previous exercises, show that $S^{-1}\mathcal{C}$ is a category, with identity morphisms given by the equivalence classes of the right fractions $X \xleftarrow{1} X \xrightarrow{1} X$.

6. Let $\mathcal{C}$ be a category, and let $S \subset \mathrm{Mor}\,\mathcal{C}$ be a class of morphisms in $\mathcal{C}$. Assume S admits a calculus of right fractions. Show that we have an *isomorphism* of categories

$$F \colon S^{-1}\mathcal{C} \to \mathcal{C}[S^{-1}]$$

given by the identity on objects, and sending a right fraction $X \xleftarrow{s} Y' \xrightarrow{\alpha} Y$ to the string $[X, \alpha, s^-, Y]$.

Hint: Note that we have a canonical functor $\mathcal{C} \to S^{-1}\mathcal{C}$ acting as identity on objects and sending a morphism $f\colon X \to Y$ to the right fraction $X \xleftarrow{1} X \xrightarrow{f} Y$. Show that this functor sends morphisms in S to isomorphisms in $S^{-1}\mathcal{C}$. Then use the universal property of $\mathcal{C}[S^{-1}]$ to get a functor $G\colon \mathcal{C}[S^{-1}] \to S^{-1}\mathcal{C}$. Show that G is the inverse to F.

7. Let $\mathcal{B}$ be an additive category, and let $S \subset \operatorname{Mor}\mathcal{B}$ be a class of morphisms in $\mathcal{C}$. Assume S admits a calculus of right fractions. Let $[s_1, \alpha_1]$ and $[s_2, \alpha_2]$ be two morphisms in $\operatorname{Hom}_{S^{-1}\mathcal{B}}(X, Y)$. In the lecture we explained how one can find a morphism $s \in S$ and morphisms β_1, β_2 in $\mathcal{B}$ such that $[s_1, \alpha_1] = [s, \beta_1]$ and $[s_2, \alpha_2] = [s, \beta_2]$. We then defined

$$[s_1, \alpha_1] + [s_2, \alpha_2] = [s, \beta_1 + \beta_2].$$

The goal of this exercise is to show that this operation is well-defined, i.e. that if $[s_1, \alpha_1] = [t, \beta_1']$ and $[s_2, \alpha_2] = [t, \beta_2']$, then

$$[s, \beta_1 + \beta_2] = [t, \beta_1' + \beta_2'].$$

In the following $[s, \alpha], [s, \alpha'], [s, \beta], [s, \beta']$ are morphisms in $\operatorname{Hom}_{S^{-1}\mathcal{B}}(X, Y)$

(a) Show that $[s, \alpha] = [s, \alpha']$ if and only if there exists some $t \in S$ such that $\alpha \circ t = \alpha' \circ t$ and $s \circ t \in S$.
Hint: Use axiom (RF3).

(b) Show that if $[s, \alpha] = [s, \alpha']$ and $[s, \beta] = [s, \beta']$, then

$$[s, \alpha + \beta] = [s, \alpha' + \beta'].$$

Hint: First show that $[s, \alpha + \beta] = [s, \alpha' + \beta]$, using part (a). Then show that $[s, \alpha' + \beta] = [s, \alpha' + \beta']$ by a similar argument.

(c) Now assume s, s_1, s_2, t and $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_1', \beta_2'$ are as above. Applying (RF2) to $X_1 \xrightarrow{s} X \xleftarrow{t} X_1'$, show that we can find $\tilde{s} \in S$ and morphisms $\gamma_1, \gamma_2, \gamma_1', \gamma_2'$ such that

$$\begin{array}{lll} [s, \beta_1] = [\tilde{s}, \gamma_1] & [s, \beta_2] = [\tilde{s}, \gamma_2] & [s, \beta_1 + \beta_2] = [\tilde{s}, \gamma_1 + \gamma_2] \\ [t, \beta_1'] = [\tilde{s}, \gamma_1'] & [t, \beta_2'] = [\tilde{s}, \gamma_2'] & [t, \beta_1' + \beta_2'] = [\tilde{s}, \gamma_1' + \gamma_2']. \end{array}$$

(d) Using part (b) and (c), conclude that $[s, \beta_1 + \beta_2] = [t, \beta_1' + \beta_2']$.

8. Let $\mathcal{B}$ be an additive category, and let $S \subset \operatorname{Mor}\mathcal{B}$ be a class of morphisms in $\mathcal{C}$. Assume S admits a calculus of right fractions. The goal of this exercise is to fill in the remaining details of the proof that $S^{-1}\mathcal{B}$ is preadditive.

(a) Show that the operation $+$ defined in the previous exercise makes $\operatorname{Hom}_{S^{-1}\mathcal{B}}(X, Y)$ into an abelian group with zero element $[1_X, 0]$.

(b) Show that composition in $S^{-1}\mathcal{B}$ is bilinear with respect to $+$.
Hint: Using axiom (RF2) three times, show that given morphisms $\alpha_1, \alpha_2\colon X \to Y$ and a morphism $s\colon Y' \to Y$, we can find morphisms $\alpha_1', \alpha_2'\colon X' \to Y'$ and a morphism $s'\colon X' \to X$ in S making the squares

$$\begin{array}{ccc} X' & \xrightarrow{\alpha_1'} & Y' \\ \downarrow{\scriptstyle s'} & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{\alpha_1} & Y \end{array} \qquad \begin{array}{ccc} X' & \xrightarrow{\alpha_2'} & Y' \\ \downarrow{\scriptstyle s'} & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{\alpha_2} & Y \end{array}$$

commutative. Use this to show that composition in $S^{-1}\mathcal{B}$ is linear in the right argument.

9. Let $X^\bullet \in \mathbf{K}(\mathcal{A})$ be a complex. Show that $X^\bullet \cong 0$ in $\mathbf{D}(\mathcal{A})$ if and only if $X^\bullet$ is exact.

10. Let $f : A^\bullet \to B^\bullet$ be a morphism in $\mathbf{K}(\mathcal{A})$. Show that the following are equivalent.

(a) f factors through an exact complex.

(b) There exists a quasi-isomorphism $q : H^\bullet \to A^\bullet$ such that $f \circ q = 0$.

(c) $f \cdot \mathrm{id}_{A^\bullet}^{-1} = 0$ in $\mathbf{D}(\mathcal{A})$.

Hint: Use the triangulated structure of $\mathbf{K}(\mathcal{A})$.