MA3204 - Exercise 4

1. (Closure properties of projectives) Let $\mathcal{A}$ be an abelian category. Show that the following hold:

   (a) The zero object in $\mathcal{A}$ is projective

   (b) If P and Q are projective in $\mathcal{A}$, then the biproduct $P \oplus Q$ is projective in $\mathcal{A}$.

   (c) If $\{P_i\}_{i \in I}$ is a collection of projective objects in $\mathcal{A}$, and if the coproduct $\coprod_{i \in I} P_i$ exists in $\mathcal{A}$, then $\coprod_{i \in I} P_i$ is projective in $\mathcal{A}$

   (d) If P is projective in $\mathcal{A}$ and $P \cong P_1 \oplus P_2$, then P_1 and P_2 are projective in $\mathcal{A}$.

2. Let $\mathbb{K}$ be a field, and let $\mathsf{Vect}_{\mathbb{K}}$ be the category of $\mathbb{K}$-vector spaces. Show that every object in $\mathsf{Vect}_{\mathbb{K}}$ is projective and injective.

3. Recall that a left R-module N is flat if $- \otimes_R N$ is an exact functor. Show that the following hold:

   - The flat R-modules satisfy the closure properties in Problem 1.
   - R is a flat left R-module.

   Conclude that any projective R-module is flat.

   Hint: First show the following

   - $- \otimes_R 0$ *is the zero functor*
   - $M \otimes_R (P \oplus Q) \cong (M \otimes_R P) \oplus (M \otimes_R Q)$.
   - $M \otimes_R (\coprod_{i \in I} P_i) \cong \coprod_{i \in I} (M \otimes_R P_i)$
   - $M \otimes_R R \cong M$.

4. Show that $\mathbb{Q}$ is flat but not projective in Ab.

   Hint: Show that there is an epimorphism from the coproduct of the family $(\mathbb{Z}/n\mathbb{Z})_{n\geq 1}$ *to* $\mathbb{Q}/\mathbb{Z}$. *Then show that the morphism* $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ *does not lift via this epimorphism*

   For flatness of $\mathbb{Q}$, *show first that any element in* $M \otimes_{\mathbb{Z}} \mathbb{Q}$ *can be written as an elementary tensor. Then show that for* $m \in M$ *and* $q \in \mathbb{Q}$ *the element* $m \otimes q$ *in* $M \otimes_{\mathbb{Z}} \mathbb{Q}$ *is* 0 *if and only if there exists an integer* k *such that* $k \cdot m = 0$. *Finally, use this to show that if* $f\colon M \to N$ *is a monomorphism, then* $f \otimes \mathrm{id}\colon M \otimes_{\mathbb{Z}} \mathbb{Q} \to N \otimes_{\mathbb{Z}} \mathbb{Q}$ *is a monomorphism*

5. Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories and $G\colon \mathcal{B} \longrightarrow \mathcal{A}$ be a right adjoint to F. Show that if G is exact, then $F(P)$ is projective for any projective object P in $\mathcal{A}$. Dually, show that if F is exact, then $G(E)$ is injective for any injective object E in $\mathcal{B}$.

   Hint: Use the adjunction isomorphism

   $$\mathrm{Hom}_{\mathcal{B}}(-, F(P)) \cong \mathrm{Hom}_{\mathcal{A}}(G(-), P)$$

   and the fact that projectivity of $F(P)$ *follows from exactness of the functor* $\mathrm{Hom}_{\mathcal{B}}(-, F(P))$.

6. Let M be a right R-module and let N be a left R-module. Show that the canonical morphism $M \times N \to M \otimes_R N$ is the universal R-balanced map with domain $M \times N$.

7. Show that

   (a) $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} Z/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$, where d is the greatest common divisor of n and m.

   (b) For any commutative ring R and any ideals I and J of R, $R/I \otimes_R R/J = R/(I + J)$.

   (c) For every right R-module M over a ring R, and every left ideal I of R, $M \otimes_R R/I = M/IM$.

   (d) $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \cong 0$.

   (e) $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}(i)$.

8. (The nine lemma) Consider the following diagram in an abelian category $\mathcal{A}$.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \xrightarrow{a_1} & A_2 & \xrightarrow{a_2} & A_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\
0 & \longrightarrow & B_1 & \xrightarrow{b_1} & B_2 & \xrightarrow{b_2} & B_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_3} & & \\
0 & \longrightarrow & C_1 & \xrightarrow{c_1} & C_2 & \xrightarrow{c_2} & C_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Assume that all the columns are exact. Using what you have learned in the lectures about exact sequences of complexes, show the following:

- If the two upper rows are exact, then the lower row is exact.
- If the two lower rows are exact, then the upper row is exact.
- If the first and third row is exact and $b_2 \circ b_1 = 0$, then the middle row is exact.

This result is typically called the nine lemma.

9. Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories. We say that F *reflects exactness* if whenever

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

is exact, then the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact. Show that if F is fully faithful and exact, then it reflects exactness.

Hint: First show that if F is exact, then it preserves images and kernels. Hence F applied to the canonical morphism $\operatorname{Im} f \to \operatorname{Ker} g$ *gives the canonical morphism* $\operatorname{Im} F(f) \to \operatorname{Ker} F(g)$. *Finally, use that F is fully faithful to show that it reflects isomorphisms, i.e. that $F(h)$ being an isomorphism implies h is an isomorphism.*