MA3204 - Exercise 4

- 1. (Closure properties of projectives) Let \mathcal{A} be an abelian category. Show that the following hold:
 - (a) The zero object in \mathcal{A} is projective
 - (b) If P and Q are projective in \mathcal{A} , then the biproduct $P \oplus Q$ is projective in \mathcal{A} .
 - (c) If $\{P_i\}_{i\in I}$ is a collection of projective objects in \mathcal{A} , and if the coproduct $\coprod_{i\in I} P_i$ exists in \mathcal{A} , then $\coprod_{i\in I} P_i$ is projective in \mathcal{A}
 - (d) If P is projective in \mathcal{A} and $P \cong P_1 \oplus P_2$, then P_1 and P_2 are projective in \mathcal{A} .
- 2. Let \mathbb{K} be a field, and let $\mathsf{Vect}_{\mathbb{K}}$ be the category of \mathbb{K} -vector spaces. Show that every object in $\mathsf{Vect}_{\mathbb{K}}$ is projective and injective.
- 3. Recall that a left R-module N is flat if $-\otimes_R N$ is an exact functor. Show that the following hold:
 - \bullet The flat R-modules satisfy the closure properties in Problem 1.
 - \bullet R is a flat left R-module.

Conclude that any projective R-module is flat.

Hint: First show the following

- $-\otimes_R 0$ is the zero functor
- $M \otimes_R (P \oplus Q) \cong (M \otimes_R P) \oplus (M \otimes_R Q)$.
- $M \otimes_R (\coprod_{i \in I} P_i) \cong \coprod_{i \in I} (M \otimes_R P_i)$
- $M \otimes_R R \cong M$.

4. Show that \mathbb{Q} is flat but not projective in Ab .

Hint: Show that there is an epimorphism from the coproduct of the family $(\mathbb{Z}/n\mathbb{Z})_{n\geq 1}$ to \mathbb{Q}/\mathbb{Z} . Then show that the morphism $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ does not lift via this epimorphism

For flatness of \mathbb{Q} , show first that any element in $M \otimes_{\mathbb{Z}} \mathbb{Q}$ can be written as an elementary tensor. Then show that for $m \in M$ and $q \in \mathbb{Q}$ the element $m \otimes q$ in $M \otimes_{\mathbb{Z}} \mathbb{Q}$ is 0 if and only if there exists an integer k such that $k \cdot m = 0$. Finally, use this to show that if $f: M \to N$ is a monomorphism, then $f \otimes \operatorname{id}: M \otimes_{\mathbb{Z}} \mathbb{Q} \to N \otimes_{\mathbb{Z}} \mathbb{Q}$ is a monomorphism

5. Let $F: \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories and $G: \mathcal{B} \longrightarrow \mathcal{A}$ be a right adjoint to F. Show that if G is exact, then F(P) is projective for any projective object P in \mathcal{A} . Dually, show that if F is exact, then G(E) is injective for any injective object E in \mathcal{B} .

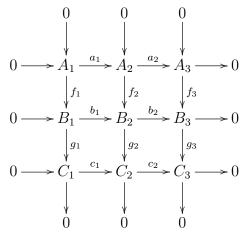
Hint: Use the adjunction isomorphism

$$\operatorname{Hom}_{\mathcal{B}}(-, F(P)) \cong \operatorname{Hom}_{\mathcal{A}}(G(-), P)$$

and the fact that projectivity of F(P) follows from exactness of the functor $\text{Hom}_{\mathcal{B}}(-, F(P))$.

- 6. Let M be a right R-module and let N be a left R-module. Show that the canonical morphism $M \times N \to M \otimes_R N$ is the universal R-balanced map with domain $M \times N$.
- 7. Show that
 - (a) $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$, where d is the greatest common divisor of n and m.
 - (b) For any commutative ring R and any ideals I and J of R, $R/I \otimes_R R/J = R/(I+J)$.
 - (c) For every right R-module M over a ring R, and every left ideal I of R, $M \otimes_R R/I = M/IM$.
 - (d) $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \cong 0$.
 - (e) $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}(i)$.

8. (The nine lemma) Consider the following diagram in an abelian category A.



Assume that all the columns are exact. Using what you have learned in the lectures about exact sequences of complexes, show the following:

- If the two upper rows are exact, then the lower row is exact.
- If the two lower rows are exact, then the upper row is exact.
- If the first and third row is exact and $b_2 \circ b_1 = 0$, then the middle row is exact.

This result is typically called the nine lemma.

9. Let $F: \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories. We say that F reflects exactness if whenever

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

is exact, then the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact. Show that if F is fully faithful and exact, then it reflects exactness.

Hint: First show that if F is exact, then it preserves images and kernels. Hence F applied to the canonical morphism $\operatorname{Im} f \to \operatorname{Ker} g$ gives the canonical morphism $\operatorname{Im} F(f) \to \operatorname{Ker} F(g)$. Finally, use that F is fully faithful to show that it reflects isomorphisms, i.e. that F(h) being an isomorphism implies h is an isomorphism.