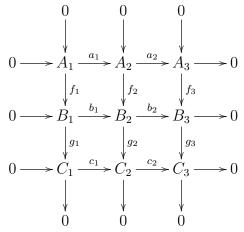
MA3204 - Exercise 4

- 1. (Closure properties of projectives) Let \mathcal{A} be an abelian category. Show that the following hold:
 - (a) The zero object in \mathcal{A} is projective
 - (b) If P and Q are projective in \mathcal{A} , then the biproduct $P \oplus Q$ is projective in \mathcal{A} .
 - (c) If $\{P_i\}_{i \in I}$ is a collection of projective objects in \mathcal{A} , and if the coproduct $\coprod_{i \in I} P_i$ exists in \mathcal{A} , then $\coprod_{i \in I} P_i$ is projective in \mathcal{A}
 - (d) If P is projective in \mathcal{A} and $P \cong P_1 \oplus P_2$, then P_1 and P_2 are projective in \mathcal{A} .
- 2. Let \mathbb{K} be a field, and let $\mathsf{Vect}_{\mathbb{K}}$ be the category of \mathbb{K} -vector spaces. Show that every object in $\mathsf{Vect}_{\mathbb{K}}$ is projective and injective.
- 3. Recall that a left *R*-module *N* is flat if $-\otimes_R N$ is an exact functor. Show that the following hold:
 - The flat *R*-modules satisfy the closure properties in Problem 1.
 - *R* is a flat left *R*-module.

Conclude that any projective R-module is flat.

- 4. Show that \mathbb{Q} is flat but not projective in Ab.
- 5. Let $F: \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories and $G: \mathcal{B} \longrightarrow \mathcal{A}$ be a right adjoint to F. Show that if G is exact, then F(P) is projective for any projective object P in \mathcal{A} . Dually, show that if F is exact, then G(E) is injective for any injective object E in \mathcal{B} .

- 6. Let M be a right R-module and let N be a left R-module. Show that the canonical morphism $M \times N \to M \otimes_R N$ is the universal R-balanced map with domain $M \times N$.
- 7. Show that
 - (a) $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} Z/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$, where d is the greatest common divisor of n and m.
 - (b) For any commutative ring R and any ideals I and J of R, $R/I \otimes_R R/J = R/(I+J)$.
 - (c) For every right *R*-module *M* over a ring *R*, and every left ideal *I* of *R*, $M \otimes_R R/I = M/IM$.
 - (d) $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} / \mathbb{Z} \cong 0.$
 - (e) $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}(i).$
- 8. (The nine lemma) Consider the following diagram in an abelian category \mathcal{A} .



Assume that all the columns are exact. Using what you have learned in the lectures about exact sequences of complexes, show the following:

- If the two upper rows are exact, then the lower row is exact.
- If the two lower rows are exact, then the upper row is exact.
- If the first and third row is exact and $b_2 \circ b_1 = 0$, then the middle row is exact.

This result is typically called the nine lemma.

9. Let $F: \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories. We say that F reflects exactness if whenever

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

is exact, then the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact. Show that if F is fully faithful and exact, then it reflects exactness.