Lecture 1

Introduction: Free resolutions, Ext and Tor

R - ring

- A module M is **free** if exists set I s.t. $M \cong \bigoplus_{i \in I} R$

M free $\iff$ M has a basis

easy to work with.

Want to measure how far a module is from being free.

Assume R noetherian, M finitely generated left R-module

Choose minimal generating set $\{m_1, \dots, m_{n_1}\}$ of M

$$F_1 = \bigoplus_{i=1}^{n_1} R \xrightarrow{f} M \qquad e_i = (0, 0, 1, 0, 0) \longmapsto m_i$$

If isomorphism ✓

If not, let $M_1 = \{x \in \bigoplus_{i=1}^{n_1} R \mid f(x) = 0\}$

M_1 gives the relations between $m_1, \dots, m_{n_1}$

repeat process on M_1 to get $\overset{F_2}{M_2}, \overset{F_3}{M_3}, \dots$

M_2 = relations between the relations of $m_1, \dots, m_n$

$M_3 = \dots$

Theorem (David Hilbert 1890)

Let $R = k[x_1, \dots, x_n]$ polynomial ring in n-variables.

(avslutles)

Then the process described above terminates after $\leq n$ steps. In other words, exists $i \leq n$ such that M_i is free. ▪

Back to arbitrary noeth ring R.

We also want to remember the F_i's and the maps relating them not just the M_i's.

$$\begin{tikzcd}
F_3 \arrow[rr, "g_3"] \arrow[dr, "f_3"'] & & F_2 \arrow[rr, "g_2"] \arrow[dr, "f_2"] & & F_1 \arrow[r, "f_1"] & M \arrow[r] & 0 \\
& M_2 \arrow[ur, hook] & & M_1 \arrow[ur, hook] & & &
\end{tikzcd}$$

the sequence (complex) $\cdots \to F_3 \xrightarrow{g_3} F_2 \xrightarrow{g_2} F_1$

contains a lot of information about M!

can be used to construct <u>abelian groups</u>

$\mathrm{Tor}_i^R(M, N)$ & $\mathrm{Ext}_R^i(M, N)$

$\forall$ right R-modules N, $\forall$ integers $i \geq 0$ $\quad$ $\forall$ left R-modules N, $\forall$ integer $i \geq 0$

central to homological algebra
measures obstructions to various constructions of M and N

Ex $R=\mathbb{Z}$

Given a subgroup A of an abelian group B, and an integer n then $\mathrm{Tor}_1^{\mathbb{Z}}(B/A, \mathbb{Z}/n)$ measures the obstruction to $nA=\{na \mid a\in A\}$ being equal to the intersection $A\cap nB$. We will see later in the course why this is the case*

$\mathrm{Tor}_i^R(M,N)$ and $\mathrm{Ext}_R^i(M,N)$

are obtained as the "homology" of "complexes" constructed from

$$\to F_3 \xrightarrow{g_3} F_2 \xrightarrow{g_2} F_1$$

Some of the main goals of the course

- Understand these words and the construction of Tor and Ext.
- Develop the necessary languge to be able to work with Tor and Ext

Course content:

1. General categories
2. Additive and abelian categories

Language for homological algebra

3. Hom and $\otimes$: Fundament for Ext and Tor
4. Complex and homology
5. Derived functors: Ext and Tor are examples of such
6. Triangulated categories

General categories

Common language to describe abstract structures in algebra, topology, geometry...

Def: A category $\mathcal{C}$ consists of:

- a class of objects $\mathrm{Ob}\,\mathcal{C}$ (class $\neq$ set)
- $\forall X, Y \in \mathrm{Ob}\,\mathcal{C}$, a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms
- $\forall X, Y, Z \in \mathrm{Ob}\,\mathcal{C}$, a map (called composition)

$$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Z)$$
$$(g, f) \longmapsto g \circ f$$

such that

- (composition has a unit) $\forall X \in \mathrm{Ob}\,\mathcal{C}$ there exists a morphism $id_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$ such that $\forall Y \in \mathrm{Ob}\,\mathcal{C}$
  - $id_X \circ f = f \quad \forall f \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$
  - $g \circ id_X = g \quad \forall g \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$

- (composition is associative) $\forall X, Y, Z, W \in \mathrm{Ob}\,\mathcal{C}$ and $\forall f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$ and $h \in \mathrm{Hom}_{\mathcal{C}}(Z,W)$ we have $(h \circ g) \circ f = h \circ (g \circ f)$

<u>Remarks</u>: We will simply write $A \in \mathcal{C}$ instead of $A \in \mathrm{Ob}\,\mathcal{C}$

- May have $\mathrm{Hom}_{\mathcal{C}}(A,B) = \emptyset$ but $\mathrm{Hom}_{\mathcal{C}}(A,A) \neq \emptyset$ since $id_A \in \mathrm{Hom}_{\mathcal{C}}(A,A)$.

- We often write $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ as an arrow $A \xrightarrow{f} B$.

<u>Ex</u>: $\mathcal{C}$ = Set: Category of sets and maps between them

- $\mathcal{C}$ = Top Category of topological spaces and continuous maps

- $\mathcal{C} = Gp$ Category of groups and group homomorphisms

- $\mathcal{C} = Ab$ Category of abelian groups and group homomorphisms

R ring

- $\mathcal{C} = \mathrm{Mod}\,R$ Category of right R-modules and R-homomorphisms

- $\mathcal{C} = \mathrm{mod}\,R$ category of finitely generated right R-modules and R-homomorphisms

- G group. Then the category $\mathcal{C}$ with one object $\bullet$ and with $\mathrm{Hom}_{\mathcal{C}}(\bullet,\bullet) = G$ is a category with composition given by multiplication

- $(X, \leq)$ poset. The <u>poset category</u> $\mathcal{C}_{(X,\leq)}$ is given by

$$\mathrm{Ob}\,\mathcal{C}_{(X,\leq)} \text{ - elements in } X$$

$$\mathrm{Hom}_{\mathcal{C}_{(X,\leq)}}(x,y) = \begin{cases} \{i_{yx}\} & x \leq y \\ \emptyset & \text{else} \end{cases}$$

with $i_{xx} = \mathrm{id}_x$ and $i_{zy} \circ i_{yx} = i_{zx}$