MA3204 - Exercise 1

1. Let $\mathcal{A}$ be an abelian category and $X^\bullet$ a complex in $\mathcal{A}$. Show the following:

   - If $H^n(X^\bullet) = 0$ for all $n < 0$, then there is a complex $Y^\bullet$ with $Y^n = 0$ for all $n < 0$ and a quasi-isomorphism $X^\bullet \longrightarrow Y^\bullet$.
   - If $H^n(X^\bullet) = 0$ for all $n > 0$, then there is a complex $Z^\bullet$ with $Z^n = 0$ for all $n > 0$ and a quasi-isomorphism $Z^\bullet \longrightarrow X^\bullet$.

   Hint: For the first claim consider $Y^n = X^n$ for $n > 0$ and choose Y^0 adequately... The second claim is proved dually.

2. Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories.

   (a) Show that F induces a functor between categories of complexes $\mathrm{Ch}(F)\colon \mathrm{Ch}(\mathcal{A}) \longrightarrow \mathrm{Ch}(\mathcal{B})$, sending a complex $X^\bullet$ to the complex obtained by applying F to each component and to each differential.

   (b) Show that the functor $\mathrm{Ch}(F)$ induces a functor

   $$\mathcal{K}(F)\colon \mathcal{K}(\mathcal{A}) \longrightarrow \mathcal{K}(\mathcal{B}).$$

3. Let P be a projective right R-module, E and injective right R-module and F a flat right R-module. Show that

   (a) $\mathrm{Ext}^n_R(P, -) = 0$ for all $n > 0$;

   (b) $\mathrm{Ext}^n_R(-, E) = 0$ for all $n > 0$;

   (c) $\mathrm{Tor}^R_n(F, -) = 0$ for all $n > 0$.

4. Show that the total complex of a double complex is a complex (i.e. the differential squares to 0).

5. Show the following result from the lecture. If R is a ring, M is a right R-module, and N is a left R-module, then we have

$$\mathrm{Tor}(M,-)(N) \cong \mathrm{Tor}(-,N)(M).$$

Hint: Use a similar argument as in the proof of the isomorphism

$$\mathrm{Ext}^i_{\mathcal{A}}(A,-)(B) \cong \mathrm{Ext}^i_{\mathcal{A}}(-,B)(A)$$

in the lecture.

6. Let B be an abelian group, let $A \subseteq B$ be a subgroup of B, and let $n > 0$ be a positive integer. In the motivation part of the first lecture we mentioned that $\mathrm{Tor}_1^{\mathbb{Z}}(B/A, \mathbb{Z}/n\mathbb{Z})$ measures the obstruction to $nA = \{n \cdot a \mid a \in A\}$ being equal to the intersection $A \cap nB$. In this exercise we investigate this problem. Prove the following:

   - The canonical inclusion $nA \to A \cap nB$ is an isomorphism if and only if the canonical map $A/nA \to B/nB$ is a monomorphism.
   - Applying the tensor product $- \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ to the exact sequence $0 \to A \to B \to B/A \to 0$, show that if $\mathrm{Tor}_1^{\mathbb{Z}}(B/A, \mathbb{Z}/n\mathbb{Z}) = 0$ then the map $nA \to A \cap nB$ is an isomorphism.

7. Consider the ring $R = \mathbb{K}[x]/\langle x^2 \rangle$, where $\mathbb{K}$ is a field.

   (a) Compute a projective resolution of the R-module $M = \langle x \rangle \subseteq R$.
   Hint: show that there exists an exact sequence

   $$0 \to M \to R \to M \to 0$$

   (b) Compute a $\mathbb{K}$-vector space basis of the vector space $\mathrm{Hom}_R(M,R)$.[1]

   (c) Compute $\mathrm{Ext}^n_R(M,M)$ for all $n \geq 0$.

   (d) Compute $\mathrm{Tor}^R_n(M,M)$ for all $n \geq 0$.

   (e) Consider the complex $Y^\bullet$ given by $Y^n = R$ for all n in $\mathbb{Z}$, with d^n_Y being the multiplication by x. Show that $Y^\bullet$ is exact. Also show that $Y^\bullet$ is not contractible (i.e. that the identity map on $Y^\bullet$ is not null-homotopic).

[1] Note that all Hom-spaces in this module category are naturally endowed with a $\mathbb{K}$-module structure...

8. Consider the ring $R = \mathbb{Z}/4\mathbb{Z}$ and consider $M = \mathbb{Z}/2\mathbb{Z}$ as an R-module (check that there is a well-defined module structure!).

   (a) Find a projective resolution of M as an R-module.
   Hint: show that there exists an exact sequence

   $$0 \to M \to R \to M \to 0$$

   (b) Compute $\operatorname{Ext}^n_R(M, M)$ for all $n \geq 0$.

   (c) Compute $\operatorname{Tor}^R_n(M, M)$ for all $n \geq 0$.

9. [1, Exercise V.2] Let R be as below, and let S be the R-module which is $\mathbb{K}$ as a $\mathbb{K}$-vector space, with all variables acting as 0. Calculate all $\operatorname{Ext}^n_R(S, S)$ for $n > 0$.

   - $R = \mathbb{K}[X]$
   - $R = \mathbb{K}[X]/(X^3)$
   - $R = \mathbb{K}[X, Y]$
   - $R = \mathbb{K}[X, Y]/(XY)$

References

[1] Steffen Oppermann, 2016 Notes in homological algebra `https://folk.ntnu.no/opperman/HomAlg.pdf`.