MA3204 - Exercise 2

1. To understand certain categorical definitions and constructions, it often helps to consider them in the special case of posets. This is what we will do in this exercise. Let $(X, \leq)$ and $(Y, \leq)$ be posets.

   (a) What is a functor $F\colon \mathcal{C}_{(X,\leq)} \to \mathcal{C}_{(Y,\leq)}$?

   (b) Given two functors $F, F'\colon \mathcal{C}_{(X,\leq)} \to \mathcal{C}_{(Y,\leq)}$, what is a natural transformation $F \to F'$?

   (c) Given two functors $F\colon \mathcal{C}_{(X,\leq)} \to \mathcal{C}_{(Y,\leq)}$ and $G\colon \mathcal{C}_{(Y,\leq)} \to \mathcal{C}_{(X,\leq)}$, what does it mean that (F, G) forms an adjoint pair?

   (d) What is a limit in $\mathcal{C}_{(X,\leq)}$? What about a colimit in $\mathcal{C}_{(X,\leq)}$?

2. Finish the proof of the Yoneda Lemma by showing that ζ^x is a natural transformation, and the associations Y and $x \mapsto \zeta^x$ are mutually inverse (see the notes).

3. Let X be a set.

   (a) If X has one element, show that $\mathsf{Hom}_{\mathsf{Set}}(X, -)\colon \mathbf{Set} \to \mathbf{Set}$ is naturally isomorphic to the identity functor.

   (b) If X has one element, show that $\mathsf{Hom}_{\mathsf{Set}}(-, X)\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ is naturally isomorphic to the constant functor which sends every set to the one-point set.

   (c) If X has two elements, show that $\mathsf{Hom}_{\mathsf{Set}}(-, X)\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ is naturally isomorphic to the powerset functor $\mathcal{P}$ given by sending a set Y to its set of subsets $\mathcal{P}(Y)$, and sending a map of sets $f\colon Y \to Y'$ to the map

$$\mathcal{P}(f)\colon \mathcal{P}(Y') \to \mathcal{P}(Y) \quad (U \subseteq Y') \mapsto (f^{-1}(U) \subseteq Y)$$

4. Finish the proof that a full, faithful and dense functor is an equivalence (see the notes).

5. Let **Ab** denote the category of abelian groups. Recall that a torsion abelian group is an abelian group where every element has finite order. In the lecture we defined

$$t\colon \mathbf{Ab} \to \mathbf{Ab}$$

by sending an abelian group M to its largest torsion subgroup $t(M)$, and sending a morphism $M \to N$ to its restriction $t(M) \to t(N)$ to the torsion subgroups.

   (a) Check that t as defined above is well-defined and gives a functor $t\colon \mathbf{Ab} \to \mathbf{Ab}$.

   (b) Check that the inclusion $t(M) \to M$ gives a natural transformation $t \to \mathrm{Id}_{\mathbf{Ab}}$

   (c) Let $\mathcal{T}$ be the full subcategory of **Ab** consisting of all torsion abelian groups. Show that t gives a functor

   $$t\colon \mathbf{Ab} \to \mathcal{T}$$

   which is right adjoint to the inclusion functor $\mathcal{T} \to \mathbf{Ab}$.

6. Show that a limit is unique up to unique isomorphism if it exists. More precisely, let $\mathcal{I}$ be a small category, let $D\colon \mathcal{I} \to \mathcal{C}$ be a functor, and let $(C, (p_i)_{i\in\mathcal{I}})$ and $(C', (p'_i)_{i\in\mathcal{I}})$ be two limits of D. Show that there exists a unique isomorphism $\phi\colon C \xrightarrow{\cong} C'$ such that the diagram

$$\begin{array}{ccc} C & \xrightarrow{\quad\phi\quad} & C' \\ & {\scriptstyle p_i}\searrow \qquad \swarrow{\scriptstyle p'_i} & \\ & D(i) & \end{array}$$

   commutes for all $i \in \mathcal{I}$.

   Similarly, colimits are unique up to unique isomorphism.

7. Let $\mathcal{I}$ be a small category, and let $\mathcal{C}$ be any category.

(a) Let $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ denote the collection of all functors $F: \mathcal{I} \to \mathcal{C}$, and for $F, G \in \mathrm{Fun}(\mathcal{I}, \mathcal{C})$ let

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{I},\mathcal{C})}(F, G)$$

denote the collection of natural transformations $F \to G$. Show that $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ is a category. This is called the *functor category* from $\mathcal{I}$ to $\mathcal{C}$. What are the isomorphisms in this category?

(b) Define $\Delta: \mathcal{C} \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$ by sending an object $C \in \mathcal{C}$ to the functor $\Delta(C): \mathcal{I} \to \mathcal{C}$ given by $\Delta(C)(i) = C$ for all $i \in \mathcal{I}$ and $\Delta(C)(f) = \mathrm{id}_C$ for all morphisms f in $\mathcal{I}$. Show that Δ can be made into a functor $\Delta: \mathcal{C} \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$.

(c) Let $D: \mathcal{I} \to \mathcal{C}$ be a functor. Show that the following data is the same:

- A limit $(\varprojlim D, (p_i)_{i \in \mathcal{I}})$ of D
- An object $\varprojlim D$ in $\mathcal{C}$ together with a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{I},\mathcal{C})}(\Delta(-), D) \cong \mathrm{Hom}_{\mathcal{C}}(-, \varprojlim D)$$

  of functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$.

Similarly, show that the following data is the same:

- A colimit $(\varinjlim D, (q_i)_{i \in \mathcal{I}})$ of D
- An object $\varinjlim D$ in $\mathcal{C}$ together with a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(\varinjlim D, -) \cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{I},\mathcal{C})}(D, \Delta(-))$$

  of functors $\mathcal{C} \to \mathbf{Set}$.

In particular, $\mathcal{C}$ has limits of shape $\mathcal{I}$ if and only if Δ has a right adjoint, and $\mathcal{C}$ has colimits of shape $\mathcal{I}$ if and only if Δ has a left adjoint.

8. [1, Exercise I.11]. Let A be an abelian group, and S and T two subgroups of A.

- Show that the pullback in $\mathbf{Ab}$ of

$$\begin{array}{ccc} & & S \\ & & \big\downarrow{\scriptstyle \mathrm{incl}} \\ T & \xrightarrow{\mathrm{incl}} & A \end{array}$$

  is given by $S \cap T$.

- Show that the pushout in **Ab** of

$$\begin{array}{ccc} A & \xrightarrow{\text{proj}} & A/S \\ \big\downarrow{\scriptstyle \text{proj}} & & \\ A/T & & \end{array}$$

  is given by $A/(S+T)$.

Do these results also hold for the category $\operatorname{Mod} R$ of modules over a ring R?

9. [1, Exercise I.12] In the category **Ab**:

- Show that the pullback

$$\begin{array}{ccc} & & L \\ & & \big\downarrow{\scriptstyle \alpha} \\ M & \xrightarrow{\beta} & N \end{array}$$

  is given by of $L \prod_N M = \{(l,m) \in L \oplus M \mid \alpha(l) = \beta(m)\}$ with the obvious maps to L and M.

- Show that the pushout of

$$\begin{array}{ccc} L & \xrightarrow{\beta} & M \\ \big\downarrow{\scriptstyle \alpha} & & \\ N & & \end{array}$$

  is given by $M \coprod_L N = M \oplus N / \{(\beta(l), -\alpha(l)) | l \in L\}$, with the obvious maps from M and N.

Do these results also hold for the category $\operatorname{Mod} R$ of modules over a ring R?

References

[1] Steffen Oppermann, 2016 Notes in homological algebra `https://folk.ntnu.no/opperman/HomAlg.pdf`.